Stabilisation in algebra, geometry, and combinatorics

Jan Draisma
TU Eindhoven and VU Amsterdam (→ Bern)

SIAM DM 16, Atlanta

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Topic 1 (Gaussian two-factor model)
$X_n := \{SS^T + D \mid S \in \mathbb{R}^{n\times 2}, D \text{ diag} > 0\}$
any $\{i_1 < \ldots < i_m\} \subseteq \{1, \ldots, n\}$ gives a map $X_n \to X_m$

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Theorem [Drton-Xiao, *Ann. Inst. Stat. Math.* 2010]
$\Sigma \in \mathbb{R}^{n\times n}$, PD, is in X_n iff all 6×6 principal submatrices are in X_6.

X_n is given by polynomial eqs and ineqs; we will focus on the eqs.

Theorem [Hilbert, *Math. Ann.* 1890]

For a field K, any ideal in $K[x_1, \dots, x_n]$ is finitely generated.

uses *Dickson's Lemma*: $\alpha_1, \alpha_2, \dots \in \mathbb{Z}^n_{\geq 0} \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$

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For every finite set S, let I_S be an ideal in $R_S := K[x_i \mid i \in S]$, such that any injection $\sigma : S \to T$ maps I_S into I_T via $x_i \mapsto x_{\sigma(i)}$. Then $I_\bullet$ is generated by $I_\emptyset, \ldots, I_{[n_0]}$ for some n_0.

$\mathrm{Sym}(S)$ *acts on* I_S*, and* $S \mapsto R_S$ *is an FI-algebra.*

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same thm for $K[x_{ij} | i \in S, j \in [k]]$ but *not* for $K[x_{ij} \mid i, j \in S]$

Topic 1, continued [Drton-Sturmfels-Sullivant, *PTRF* 2007]

$X_n \subseteq \mathbb{R}^{n\times n}$ 2-factor model, vanishing ideal $I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]]$

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$x_{ij} - x_{ji} \in I_n$ for $n \geq 2$
off-diagonal 3×3-subdeterminants $\in I_n$ for $n \geq 6$
$\sum_{\pi\in\mathrm{Sym}(n)} \mathrm{sgn}(\pi)\pi \cdot x_{12}x_{23}x_{34}x_{45}x_{51} \in I_n$ for $n \geq 5$

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Replacing 2 *by k we know only weaker stabilisation:*

Theorem [D, *Adv. Math.* 2010]

$\forall k\ \exists n_0$ such that via injections $[n_0] \to [n]$ the ideal I_{n_0} generates I_n *up to radical.*

Instances of stabilisation

(using *Noetherianity up to symmetry*)

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

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k	0	1	2	3	4
d	1	2	$3^\dagger$	$4^\bullet$	$\geq 9^*$

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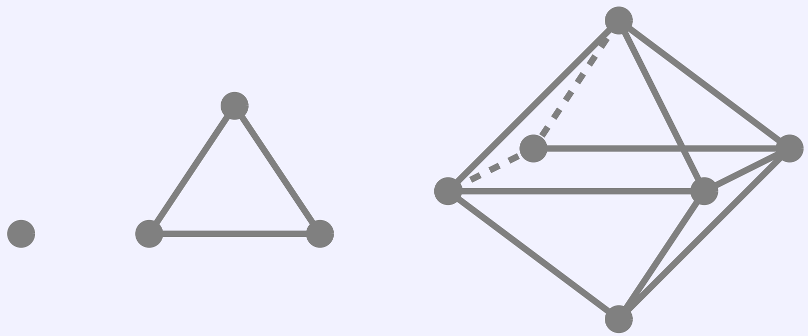
Snowden has a stabilisation result for higher syzygies for $k = 1$.

Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



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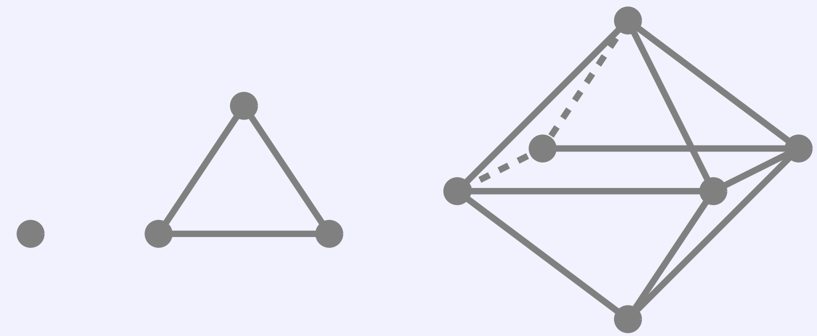
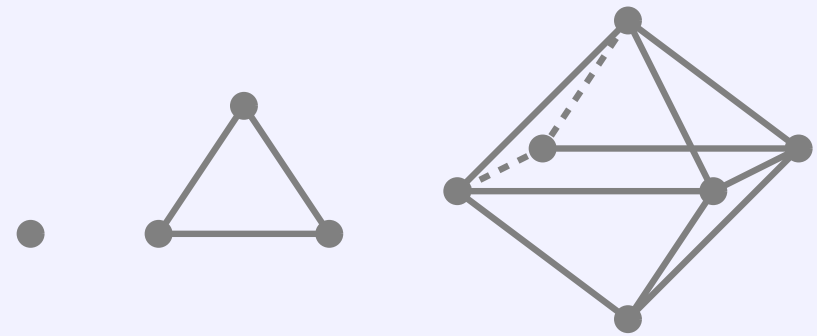


Theorem [De Loera-Sturmfels-Thomas, *Combinatorica* 1995]

P_n has a Markov basis consisting of *moves* $v_{ij} + v_{kl} \to v_{il} + v_{kj}$ and $v_{ij} \to v_{ji}$ for i, j, k, l distinct; i.e., if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$, then the expressions are connected by such moves.



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Theorem [D-Eggermont-Krone-Leykin *Algebra & Number Th* 2016]

Any sequence $(P_n \subseteq \mathbb{Z}^n)_n$ of lattice point configurations such that $P_n = \mathrm{Sym}(n) P_{n-1}$ for $n \gg 0$ admits a sequence $(M_n)_n$ of Markov bases such that $M_n = \mathrm{Sym}(n) M_{n-1}$ for $n \gg 0$.

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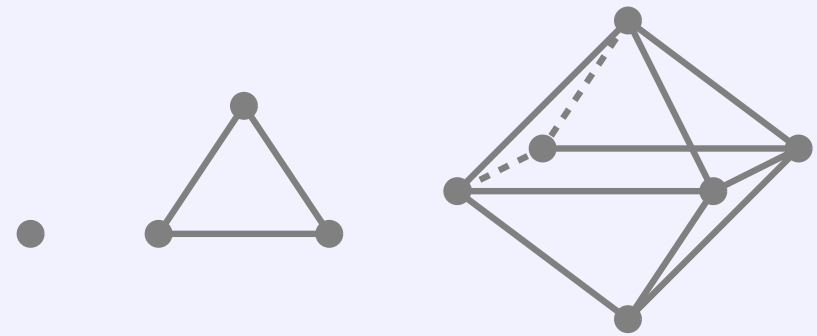

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(Also true for $P_n \subseteq \mathbb{Z}^{k \times n}$, considered a subset of $\mathbb{Z}^{k \times (n+1)}$ by adding a zero column. We also have an algorithm for computing $(M_n)_n$.)

Topic 4: homological stability

M a compact manifold
for a finite set S define $C_S(M) := \{(p_i)_{i\in S} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq M^S$
for any injection $S \subseteq T$ have map $C_T(M) \to C_S(M)$
dually: $H^d(C_S(M), \mathbb{Q}) \to H^d(C_T(M), \mathbb{Q})$.

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Among other things, this implies that the $\mathrm{Sym}(S)$*-character of* $H^d(C_S(M), \mathbb{Q})$ *is constant for* $|S| \gg 0$.

Topic 5: speculating on matroid theory

Fix field K. For a finite set S and a natural number d let $X_{d,E} \subseteq \mathrm{Gr}(d, K^E)$ be a *Zariski-closed* subset, such that:

1. deletion $\mathrm{Gr}(d, K^E) \to \mathrm{Gr}(d, K^{E-i})$ maps $X_{d,E}$ into $X_{d,E-i}$;
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Remark
For *finite* K this is the matroid minor theorem (Geelen-Gerards-Whittle).
For *infinite* K, the MMT does not hold, but the above might.

Grassmannians

$\mathrm{Gr}_k(V)$ is a variety parameterising k-dimensional subspaces of V. It is functorial in V, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$.

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A sequence $X_0, X_1, X_2, \ldots$ of rules $V \mapsto X_k(V) \subseteq \bigwedge^k(V)$ satisfying these two properties is called a *Plücker variety*.

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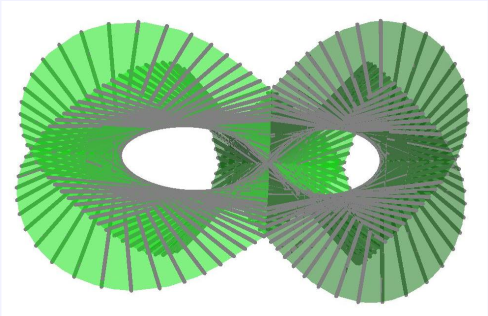

Construction of new Plücker varieties

tangential variety, secant variety, etc.

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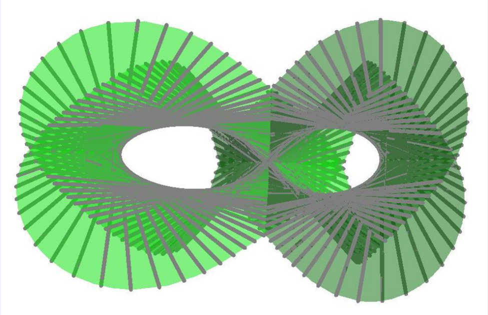

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Theorem [D-Eggermont *Crelle* 201?]

For *bounded* Plücker varieties, $(X_k(K^n))_{k,n-k}$ stabilises.

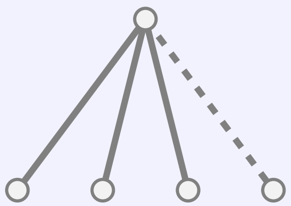
(*For* $X = \mathrm{Gr}$, X_∞ = *Sato's Grassmannian* $\subseteq$ *dual infinite wedge.*)

Stabilisation in other areas

Algebraic statistics

families of graphical models where the graph grows

[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont,. . .]



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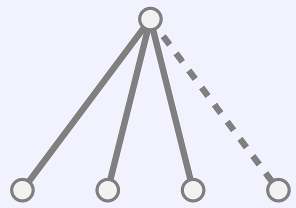

Commutative algebra and representation theory

higher syzygies, sequences of modules

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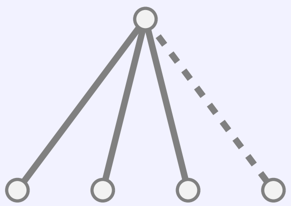

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