Algebra and Geometry of Tensors 2: Structured Tensors

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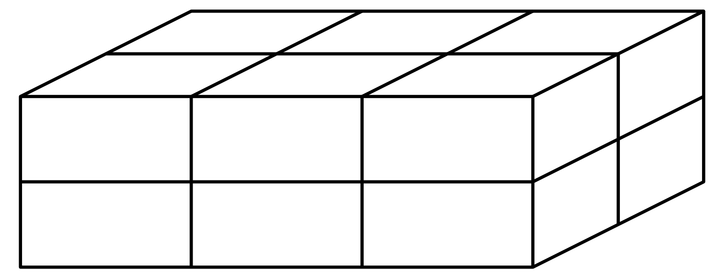
(Structured) tensors

Definition. An *order-d* tensor is an $n_1 \times \cdots \times n_d$ block of $\mathbb{C}$-numbers; or: a $T \in V_1 \otimes \cdots \otimes V_d$, where $\dim V_k = n_k$.

a

a_1
a_2

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}



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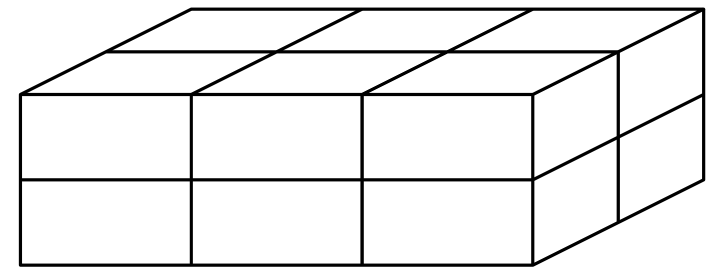

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Structure: usually an interesting subspace of tensors, e.g.:
- *symmetric tensors* (homogeneous polynomials): $V_i = V$, $S^d(V) = \{T \mid \forall \sigma \in S_d : \sigma(T) = T\}$ (GL(V) acts);
- *partially symmetric tensors*: e.g. $S^2\mathbb{C}^2 \otimes \mathbb{C}^3$ ($\mathrm{GL}_2 \times \mathrm{GL}_3$);
- a single interesting tensor (symmetry?)

The simplest tensors

Fact: W an irreducible representation of a connected algebraic group G, then $\mathbb{P}W$ has a unique minimal G-orbit X.

• $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)$, $W = V_1 \otimes \cdots \otimes V_d \Rightarrow \widehat{X} = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in V_i\}$; X is *Segre embedding* of $\prod_i \mathbb{P}V_i$
• $G = \mathrm{GL}(V)$, $W = S^d V \Rightarrow \widehat{X} = \{v \otimes \cdots \otimes v \mid v \in V\}$; X is *Veronese embedding* of $\mathbb{P}(V)$
• $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$, $W = S^{d_1} V_1 \otimes S^{d_2} V_2 \Rightarrow \widehat{X} = \{v_1^{\otimes d_1} \otimes v_2^{\otimes d_2}\}$; X is *Segre-Veronese embedding*, etc.

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Definition: • $\sigma_k^0 \widehat{X} = \{w_1 + \cdots + w_k \mid w_i \in \widehat{X}\}$
• $\sigma_k \widehat{X} = \overline{\sigma_k^0 \widehat{X}} = \widehat{\sigma_k X}$ the k-th *secant variety* of $\widehat{X}$
• $\mathrm{rk}_X(T) = \min\{k : T \in \sigma_k^0 \widehat{X}\}$,
$\mathrm{brk}_X(T) = \min\{k : T \in \sigma_k \widehat{X}\}$—*rank* and *border rank*

Examples/remarks

• $n_1 \times n_2$-matrices: $\widehat{X}$ is the variety of rank ≤ 1 matrices, $\sigma_k^0 \widehat{X} = \sigma_k \widehat{X}$ the variety of (ordinary) rk $\leq k$ matrices. A decomposition of T of rank k corresponds to a factorisation $T = A \cdot B$ with $A \in \mathbb{C}^{n_1 \times k}$, $B \in \mathbb{C}^{k \times n_2}$

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• the 3×3-permanent $\sum_{\pi \in S_3} x_{1,\pi(1)}x_{2,\pi(2)}x_{3,\pi(3)} \in S^3(\mathbb{C}^9)$ has rank 16 (Shitov) and border rank 16 (Conner-Huang-Landsberg) $\rightsquigarrow$ Amy Huang's talk.

(Non-)defectiveness

Definition: We expect $\dim \sigma_k \widehat{X} = \min\{\dim W, k \dim \widehat{X}\}$; otherwise, $\sigma_k \widehat{X}$ is *defective*.

Remark: For $d = 2$ (matrices), almost always defective, due to $A \cdot B = (A \cdot g) \cdot (g^{-1} \cdot B)$.

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Some known facts

• $\exists$ very few defective secant varieties for $\widehat{X} \subseteq S^d \mathbb{C}^n$ (Veronese): all cases with $d = 2$ plus four cases with $d \in \{3, 4\}$. (Alexander-Hirschowitz)

• conjecture for secant varieties for Segre embeddings (Abo-Ottaviani-Peterson, work by many)

• $\widehat{X} \subseteq S^{d_1} V_1 \otimes S^{d_2} V_2$ has non defective $\sigma_k \widehat{X}$ for $d_1, d_2 \geq 3$ (Galuppi-Oneto $\rightsquigarrow$ Francesco Galuppi's talk)

(Non-)uniqueness of decompositions

Assume $\dim \sigma_k \widehat{X} = k \dim \widehat{X}$. Then a sufficiently general $T \in \sigma_k \widehat{X}$ has a finite number ℓ of decompositions into k terms.

Theorem

For $\widehat{X} = \{v^{\otimes d} \mid v \in V\} \subseteq S^d V$:

• if in addition $k \dim \widehat{X} < \dim S^d V \Rightarrow \ell = 1$, except in three cases, where $\ell = 2$ (Chiantini-Ottaviani-Vannieuwenhoven)

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$\rightsquigarrow$ Luca Chiantini's talk: reducing a decomposition of $T \in S^d V$ with too many terms to one in $\mathrm{rk}(T)$ terms.

$\rightsquigarrow$ Elena Angelini's talk: specific ternary forms and their identifiability, in particular of degree 9.

Remark
$\sigma_k X = \overline{\bigcup_R \langle R \rangle}$ where R runs through all k-element subsets of $X \subseteq \mathbb{P}W$.

Definition: The k-th *cactus variety* of X is $\overline{\bigcup_R R}$ where the union is over all length-k subschemes of X. (Buczyńska-Buczyński, Kanev-Iarrobino)

Since for moderately large k, not every length-k is the limit of schemes of k reduced points, $\sigma_k X \subsetneq$ k-th cactus variety. Using this, B-B showed: many secant varieties of Veronese are *not* defined by minors of catalecticant matrices.

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$\rightsquigarrow$ Jarek Buczyński's talk: the cactus varieties to sufficiently ample embeddings of X *are* defined by minors of certain matrices of linear forms.

Apolarity

Already saw $S^e V^* \times S^d V \to S^{d-e} V$ —*apolarity action*; for $T \in S^d V$, $T^0 \subseteq SV^*$ is an ideal.

Apolarity lemma: $T \in S^d V$ admits a decomposition as $c_1 v_1^{\otimes d} + \cdots + c_k v_k^{\otimes d}$ iff the vanishing ideal of $[v_1], \ldots, [v_k] \in \mathbb{P}V$ is contained in $T^0 \subseteq SV^*$.

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Recently extended to alternating tensors (Arrondo-Bernardi-Macias Marques-Mourrain).

$\rightsquigarrow$ Reynaldo Staffolani's talk: a version of apolarity for the minimal orbit X in $\mathbb{P}W$, W any irreducible $\mathrm{GL}(V)$-representation.

Other rank notions

Matrix rank generalises to tensors in *many* other ways!

Definition

The *multilinear/Tucker rank* of $T \in V_1 \otimes \cdots \otimes V_d$ is $(\dim U_1, \ldots, \dim U_d) \in \mathbb{N}^d$ where $U_i \subseteq V_i$ is the image of the corresponding flattening $T : \bigotimes_{j \neq i} V_j^* \to V_i$.

$\rightsquigarrow$ Khazhgali Kozhasov's talk: over $\mathbb{R}$, the tensors of fixed multilinear rank form a *minimal manifold* (Heaton-Kozhasov-Venturello)

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The *slice rank* of $T \in V_1 \otimes \cdots \otimes V_d$ is $\min\{k : T \in \sigma_k^0 Y\}$ where $Y = \bigcup_{i=1}^d \{v \otimes S \mid v \in V_i, S \in \bigotimes_{j \neq i} V_j\}$

(Tao's description of Ellenberg-Gijswijt's resolution of the cap set problem)

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Definition
The *geometric rank* of $T \in V_1 \otimes V_2 \otimes V_3$ is $\operatorname{codim}_{V_1^* \times V_2^*} \{(x, y) \mid T(x \otimes y) = 0 \in V_3\}$

Example: If $V_1 = V_2 = V_3 = V$ of dimension $n > 3$ over a quasi-algebraically closed field K, $T \in \bigwedge^3 V$ is *alternating*, then $\exists$ linearly independent $x, y \in V^*$ such that $T(x \otimes y) = 0$ (Draisma-Shaw). The geometric rank is generically $n - 2$ (for n even) respectively $n - 1$ (for n odd).

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$\rightsquigarrow$ Jeroen Zuiddam's talk geometric rank $\leq$ slice rank
(Kopparty, Moshkovitz, Zuiddam)

Tensors over $\mathbb{R}$

Unlike real matrices, a *real* tensor T with $d \geq 3$ can have $\mathrm{rk}_{\mathbb{R}}(T) > \mathrm{rk}_{\mathbb{C}}(T)$.

Definition

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Theorem

The typical ranks form an interval of integers, starting with the generic rank over $\mathbb{C}$. (Bernardi-Blekherman-Ottaviani)

$\rightsquigarrow$ Chiara Brambilla's talk: for binary forms, algebraic boundaries between the various typical ranks are unions of dual varieties of coincident root loci.

Enter the orthogonal group

still work over $\mathbb{R}$, and equip each V_i with an inner product

Definition: $T \in V_1 \otimes \cdots \otimes V_d$ is *odeco* if $T = \sum_{i=1}^{k} \lambda_i v_{i,1} \otimes \cdots \otimes v_{i,d}$ where $\forall j : v_{1,j}, \ldots, v_{k,j}$ orthonormal in V_j. (Variant for $S^d V$).

If all $V_i = V$ of dimension n, these form a closed semi-algebraic set of dimension $n + d \cdot (n(n-1)/2)$, defined by quadratic equations (Boralevi-D-Horobeţ-Robeva).

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$\rightsquigarrow$ Konstantin Usevich's talk: decomposition of symmetric tensors T as $\sum_i \lambda_i v_i^{\otimes d}$, $V = (v_1 | \ldots | v_k)$ norm-1 columns $VV^T = \frac{k}{n} I_n$ (conjecture by Oeding-Robeva-Sturmfels).

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