Tropical geometry, secant varieties, and multivariate interpolation

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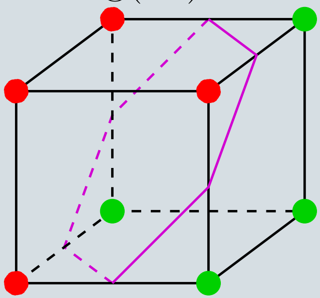
Slicing cubes

Conjecture. *The n–cube $\{0,1\}^n$ is* sliceable *for all $n \neq 4$.*

$\Rightarrow \mathrm{Seg}(\mathbb{P}^1)^n$ non–defective



true for $n \leq 10$ (Immanuel Halupczok)

Definition. A *slice* of $S \subseteq \mathbb{R}^n$ is $P \subseteq S$ with
$\dim \mathrm{Aff}\, P = n$,
P affinely independent, and
P separable from $S \setminus P$ by a hyperplane.

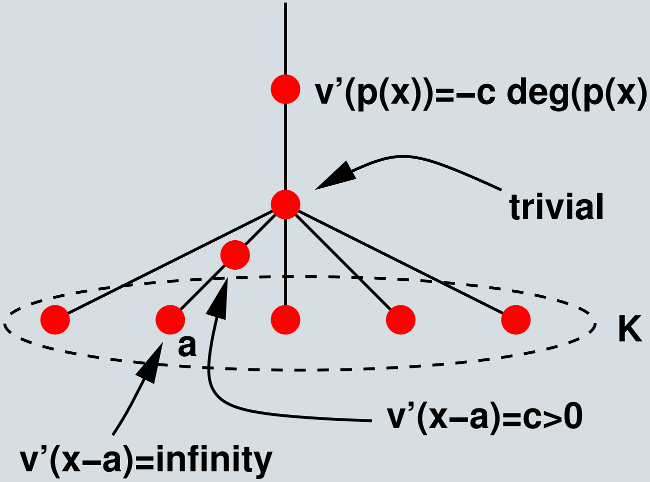
S is *sliceable* if
S is affinely independent or
S has a slice P with $S \setminus P$ sliceable.

Berkovich spaces

K a field with non–archimedean valuation $v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$
X affine variety over K

Definition. $\mathcal{T}X := \{v' : K[X] \to \overline{\mathbb{R}} \mid v'$ ring valuation extending $v\}$
Berkovich space

Example. K algebraically closed, v trivial, $X = \mathbb{A}^1$:



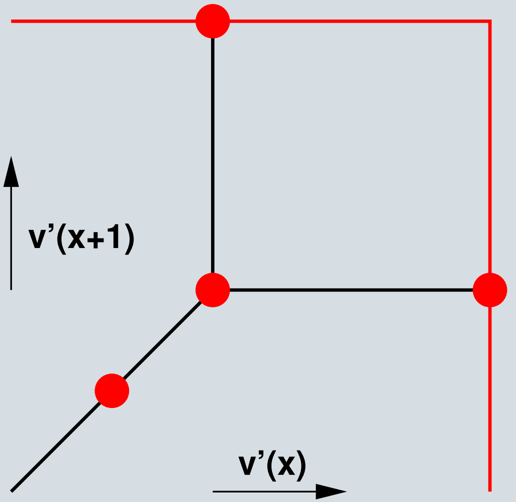
v non–trivial: complicated tree, locally like this one ($\rightsquigarrow$ Matt Baker's lectures on the web)

Tropical geometry

$x := (x_1, \dots, x_n)$ generators of $K[X] \rightsquigarrow$ embedding $X \hookrightarrow \mathbb{A}^n$

Definition. $\mathcal{T}_x X := \{(v'(x_1), \dots, v'(x_n)) \mid v' \in \mathcal{T}X\} \subseteq \overline{\mathbb{R}}^n$ *tropicalisation of X w.r.t. x.*

Example. $X = \mathbb{A}^1$, $K[X] = K[x]$, $x_1 = x$, $x_2 = x + 1$:



Remark. $\mathcal{T}X = \lim_{\leftarrow} \mathcal{T}_x X$.

Alternative characterisations

(L, v) valued extension of K
$\rightsquigarrow X(L) \to \mathcal{T}X,\ p \mapsto (f \mapsto v(f(p)))$

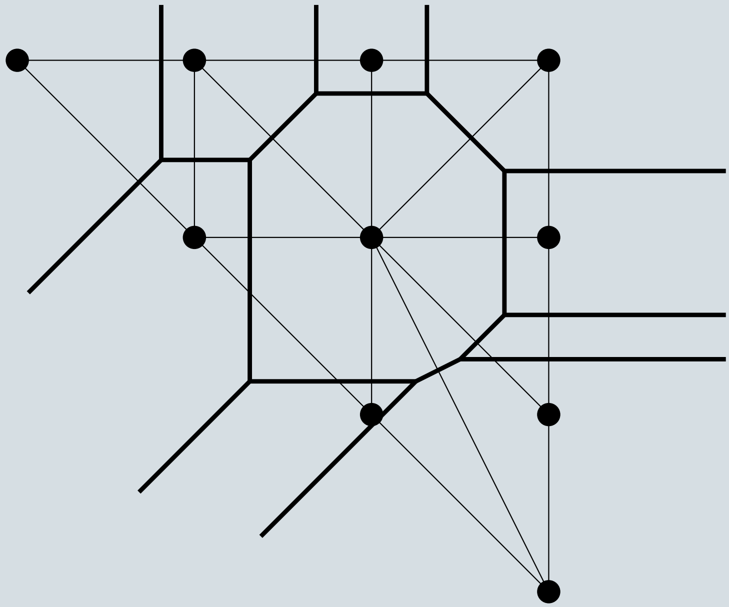
Definition. $f = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \dots, x_n]$
$\rightsquigarrow \mathcal{T}f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}, w \mapsto \min_\alpha v(c_\alpha) + \langle w, \alpha \rangle$ *tropicalisation of* f.

Theorem (Einsiedler–Kapranov–Lind, Speyer–Sturmfels, Payne, D, ...).
For suitable L*:*

image of $X(L) \to \mathcal{T}_x X =$
$\mathcal{T}_x X =$
$\{w \in \overline{\mathbb{R}}^n \mid \forall f \in I(X) : \mathcal{T}f$ *not differentiable at* $w\}$.

Curve counting

Example. a tropical elliptic curve:



major applications in *curve counting* (Mikhalkin, Gathmann–Markwig, Brugallé, Bertrand, ...)

Fundamental theorems

Theorem (Bieri–Groves). $\mathcal{T}_x X$ *is polyhedral complex of dimension* $\dim X$.

Theorem (Bogart–Jensen–Speyer–Sturmfels–Thomas, Hept–Theobald). $\exists$ *finite* $B \subseteq I(X)$ *with*
$\mathcal{T} X = \{w \in \overline{\mathbb{R}}^n \mid \forall f \in B : \mathcal{T} f$ *not differentiable at* $w\}$
(tropical basis)

Remark. B can be taken of size $n+1$ at the cost of high degrees.
But $B = \{\text{all circuits}\}$ for a *linear* tropical basis of a linear space.

Application to secant varieties

$f : K^m \to C \subseteq K^n$, closed cone
$f^{(k)} : (K^m)^k \to kC, (v_1, \ldots, v_k) \mapsto f(v_1) + \ldots + f(v_k)$

Lemma. $\mathcal{T} f^{(k)} : (\mathbb{R}^m)^k \to \mathcal{T}_x(kC) \subseteq \mathbb{R}^n$

$\mathcal{T} f^{(k)}$ piecewise linear

Proposition. $\dim kC \geq \operatorname{rk} d_w \mathcal{T} f^{(k)}$

Method for proving good lower bounds:

1. choose a *good parameterisation* f, and
2. *maximise* the right–hand side over w.

Good parameterisations

G semisimple algebraic group
V irreducible G–module
C orbit of highest weight vectors

Example. Segre–Veronese embeddings, Grassmannians, flag varieties, etc.

$v_0 \in C$
$P = \mathrm{Stab}_G K v_0 \supseteq T$ maximal torus
$\bigoplus_{i=1}^{m-1} KY_i$ is the T–stable complement of $\mathrm{Lie}(P) \subseteq \mathrm{Lie}(G)$.

$$f : K^m \to C,\ (t_1, \ldots, t_m) := t_m \cdot \exp(t_1 Y_1) \cdots \exp(t_{m-1} Y_{m-1}) v_0.$$

Equip V with suitable PBW–type coordinates x_i.

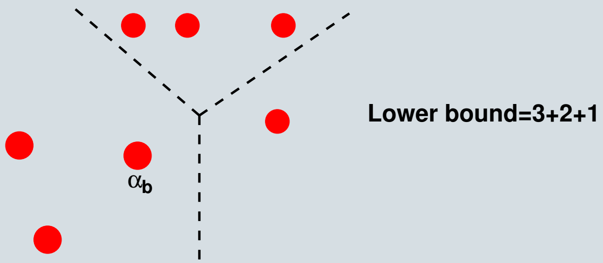
Theorem (Baur–D). *$\mathcal{T} f$ hits a full–dimensional subset of $\mathcal{T}_x C$.*

What about $\mathcal{T} f^{(k)}$?

Maximisation problem

Recall:
$f : K^m \to C \subseteq K^n$
$f^{(k)} : (K^m)^k \to kC$
$\dim kC \geq \operatorname{rk} d_w \mathcal{T} f^{(k)}$
$f =: (f_1, \ldots, f_n)$

Theorem (D). *If* $f_b = c_b t^{\alpha_b}$ (toric case) *then*
$\dim kC \geq \max_{C_1,\ldots,C_k} \sum_{i=1}^{k} (1 + \dim \operatorname{Aff}\{\alpha_b | b = 1, \ldots, n \textit{ and } \alpha_b \in C_i\})$
over all regular subdivisions of $\mathbb{R}^m$ *into* $C_1, \ldots, C_m$.



Back to sliceability

Corollary. *If $S := \{\alpha_1, \ldots, \alpha_b\} \subseteq \operatorname{Aff} S$ sliceable then C non–defective.*

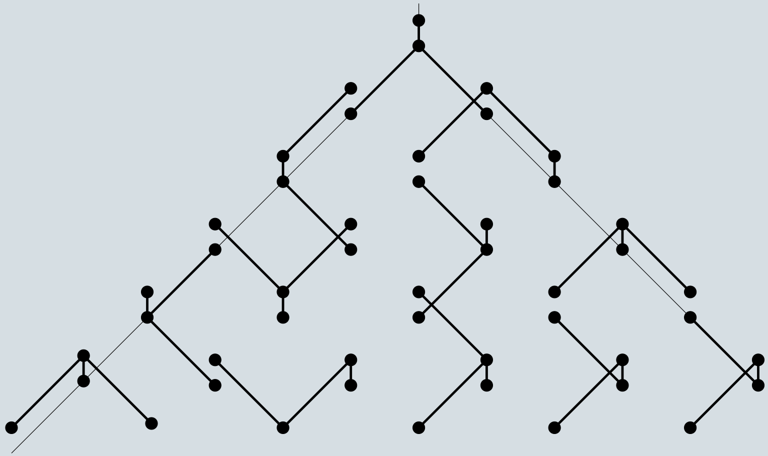
Remark. I know no toric f for which the tropical bound is not sharp.

Some toric results: secant dimensions of

1. Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$ (Catalisano–Geramita–Gimigliano)
2. S–V of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$ (CGC)
3. S–V embeddings of $\mathbb{P}^1 \times \mathbb{P}^2$
4. V embeddings of $\mathbb{P}^3$ (Alexander–Hirschowitz, Brannetti)
5. Segre embeddings of $(\mathbb{P}^1)^n$ for $n \leq 10$ (Halupczok)

First non–toric results

1. all SL_3–equivariant embeddings of $\{(p,l) \mid p \subseteq l \subseteq \mathbb{P}^2\}$ (Baur–D)



2. (after a question of Geramita:)
   partition $3 \leq d = d_1 + \ldots + d_p$
   $C := \{f_1 \cdots f_p \mid \deg f_t = d_t\} \subseteq K[x_1, \ldots, x_{2k}]_d$
   kC is non–defective

Some literature

Sturmfels and Sullivant
Combinatorial secant varieties Pure Appl. Math. Q. 2(3):867–891, 2006.

Ciliberto, Dumitrescu, and Miranda
Degenerations of the Veronese and Applications
Preprint, 2007

Develin, *Tropical secant varieties of linear spaces*
Discrete Comput. Geom. 35(1): 117–129, 2006.

D., *A tropical approach to secant dimensions*
Pure Appl. Algebra 212:349–363, 2008 (also on arxiv).

Baur and D. *Secant dimensions of low–dimensional homogeneous varieties*
Preprint, 2007, on arxiv.

And now for sg. completely different..

Question. $\exists ? f \in \mathbb{C}[x]$ *with*
degree $n > 0$*,*
$f(0) = f(1) = 0$*, and*
$\gcd(f, f^{(k)}) \neq 1$ *for all* $k = 1, \ldots, n-1$*?*

No–one knows!

Proposition 1 (Exercise!). *Fix* $n, k_0 \in \{1, \ldots, n-1\}$.
$\exists f \in \mathbb{R}[x]$ *with*
$f(0) = f(1) = 0$*,*
only real roots*, and*
$\gcd(f, f^{(k)}) \neq 1$ *for all* $k = 1, \ldots, \hat{k}_0, \ldots, n-1$.