Higher Secants of Sato's Grassmannian

Jan Draisma
TU Eindhoven

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Grassmannians: functoriality and duality

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$\rightsquigarrow \mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
cone over Grassmannian
(*rank-one alternating tensors*)

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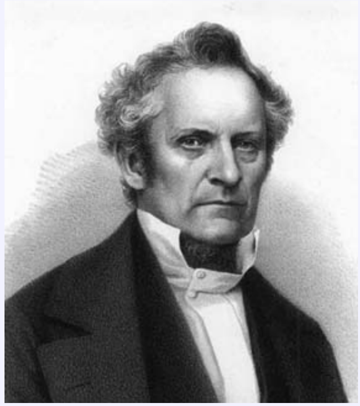
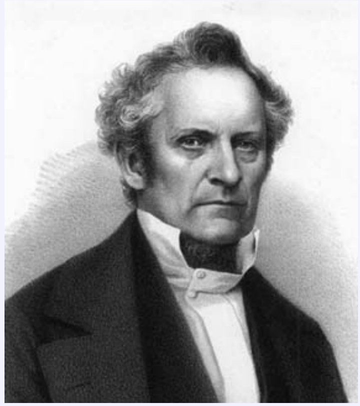
2. if $\dim V =: n + p$ with $n, p \geq 0$
$\rightsquigarrow$ natural map $\bigwedge^p V \to (\bigwedge^n V)^* \to \bigwedge^n(V^*)$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_n(V^*)$

Plücker varieties



Definition
Rules $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ with

$$\mathbf{X}_p : \{\text{vector spaces } V\} \to \{\text{varieties in } \textstyle\bigwedge^p V\}$$



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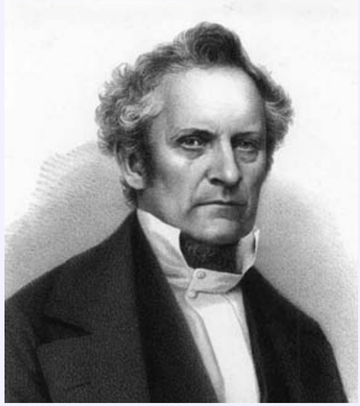

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Constructions
$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are
$\mathbf{X} + \mathbf{Y}$ *(join)*, $\tau\mathbf{X}$ *(tangential)*,
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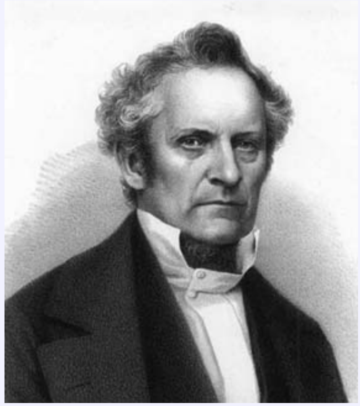

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skew analogue of Snowden's Δ-varieties

Results, with Eggermont

Definition

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Theorems apply, in particular, to
$k\mathbf{Gr}$ = {alternating tensors of *alternating rank* $\leq k$}

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Lots of applications: algebraic statistics, multilinear algebra, …but not needed today.

Topological Noetherianity up to symmetry

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A topological space X with a group G acting on it is *G-Noetherian* if every chain $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of G-stable closed subsets stabilises.

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Constructions

• G-stable subsets, and G-equivariant images, and finite unions of G-Noetherian spaces are G-Noetherian.
• If G acts on X and $Y \subseteq X$ is H-Noetherian for some subgroup $H \subseteq G$, then GY is G-Noetherian.

Theorem
Set $\mathrm{GL}_{\mathbb{N}} := \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(K)$ and $M_{\mathbb{N}} := K^{\mathbb{N}\times\mathbb{N}}$. For any $N \in \mathbb{N}$, $M_{\mathbb{N}}^N$ is $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian with the Zariski topology.

Tuples of infinite-by-infinite matrices

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Key notion
$A_1, \ldots, A_N$ matrices of same sizes (perhaps infinite)
$\rightsquigarrow \mathrm{rk}(A_1, \ldots, A_N) := \min\left\{\mathrm{rk}\left(\sum c_i A_i\right) \mid (c_1 : \ldots : c_N) \in \mathbb{P}^{N-1}\right\}$

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2nd case: $X_n = M_{\mathbb{N}}^N$ for all n □

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Back to Plücker varieties

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Approach: organise all $\mathbf{X}_p(V)$ *into one infinite-dimensional space.*

The infinite wedge

$$V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle_K$$
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Definition

$\bigwedge^{\infty/2} V_\infty := \lim_{\rightarrow} \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \ldots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

Young diagrams

Recall

$\bigwedge^{\infty/2} V_\infty$ has basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, where
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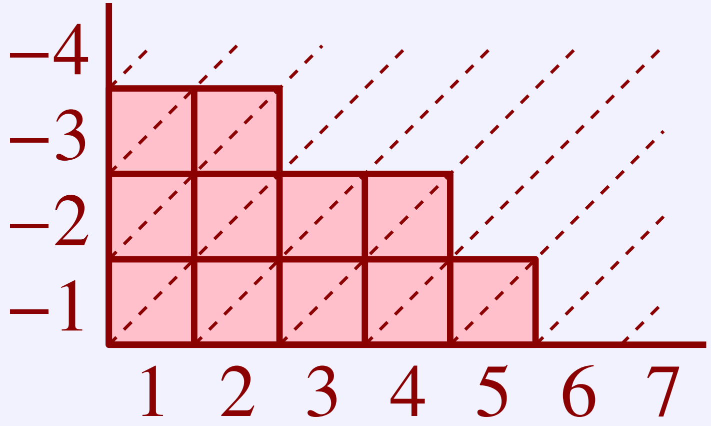
Bijection with Young diagrams

x_I with $I = \{-3, -2, 1, 2, 4, 6, 7, \ldots\}$ corresponds to



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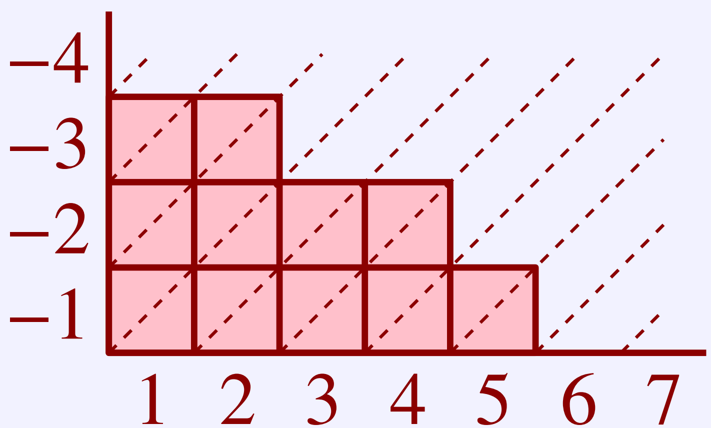

These x_I will be the *coordinates* of our ambient space, partially ordered by $I \leq J$ if $i_k \geq j_k$ for all k (inclusion of Young diags).
Unique minimum is $I = \{1, 2, \ldots\}$.

The limit of a Plücker variety

Dual diagram

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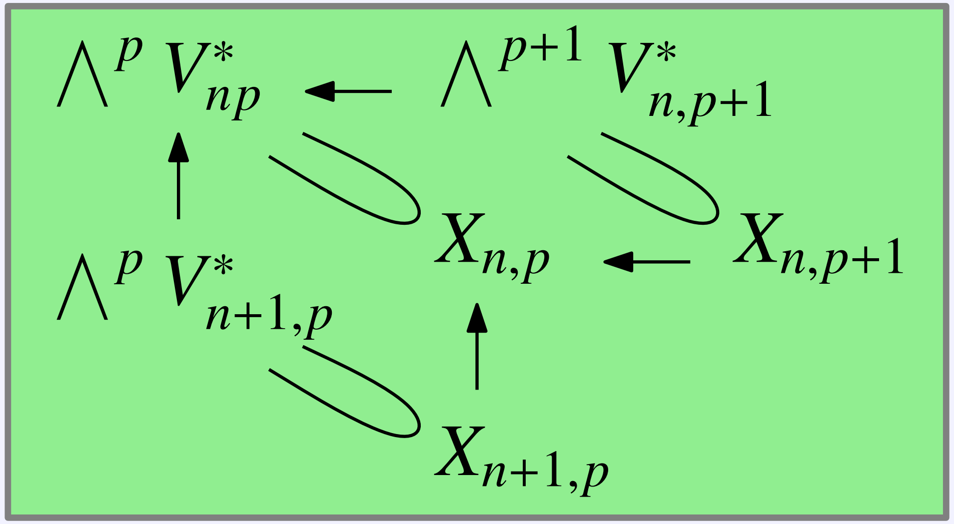

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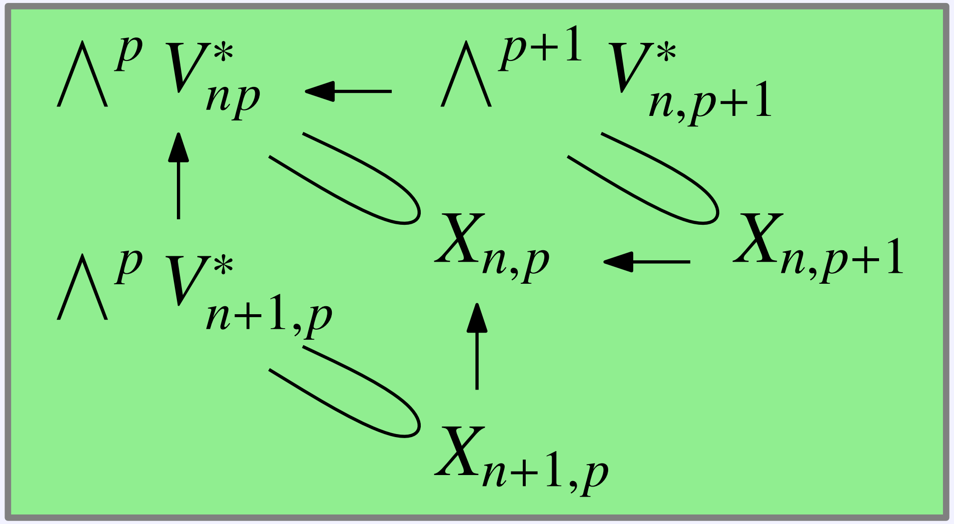
\end{array}
$$



$\{\mathbf{X}_p\}_{p\geq 0}$ a Plücker variety $\rightsquigarrow$ varieties $X_{n,p} := \mathbf{X}_p(V_{n,p}^*)$

$\rightsquigarrow$ $\mathbf{X}_\infty := \lim_{\leftarrow} X_{n,p}$ is GL_∞-stable subvariety of $(\bigwedge^{\infty/2} V_\infty)^*$

Theorem (implies Main Theorem)
For *bounded* $\mathbf{X}$, the limit $\mathbf{X}_\infty$ is cut out by finitely many GL_∞-orbits of equations.

Example

The limit $\mathbf{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$ of $(\mathbf{Gr}_p)_p$ is *Sato's Grassmannian* defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$
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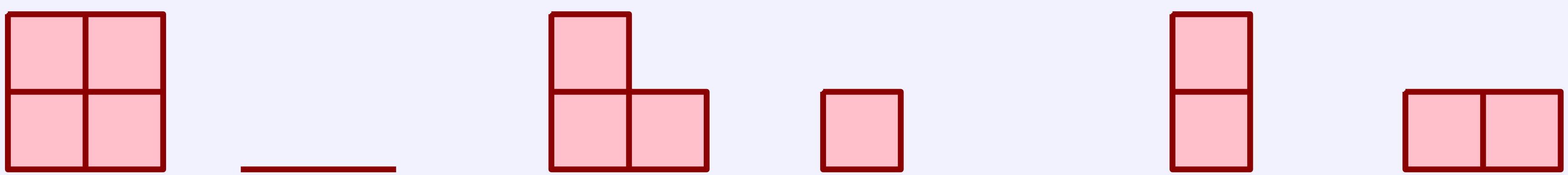

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Sato's Grassmannian

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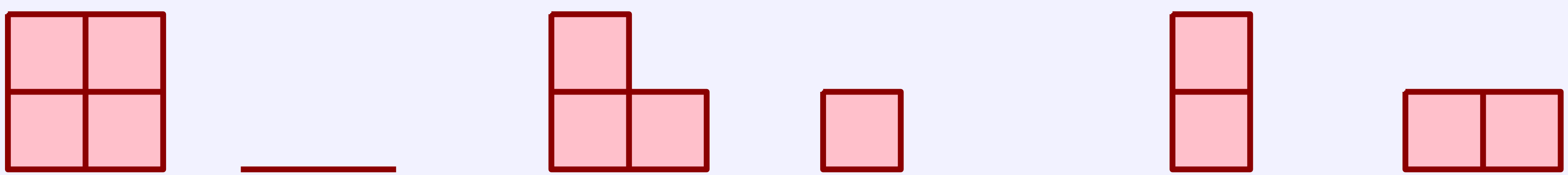

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Our theorems imply that also higher secant varieties of Sato's Grassmannian are defined by finitely many GL_∞-orbits of equations. . . *we just don't know which!*

Setting

$\mathbf{X}$ bounded Plücker variety $\rightsquigarrow \exists\, n_0, p_0$ such that GL_∞-orbits of equations of $X_{n_0,p_0} \subseteq \bigwedge^{p_0} V^*_{n_0,p_0}$ define $\mathbf{X}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$.

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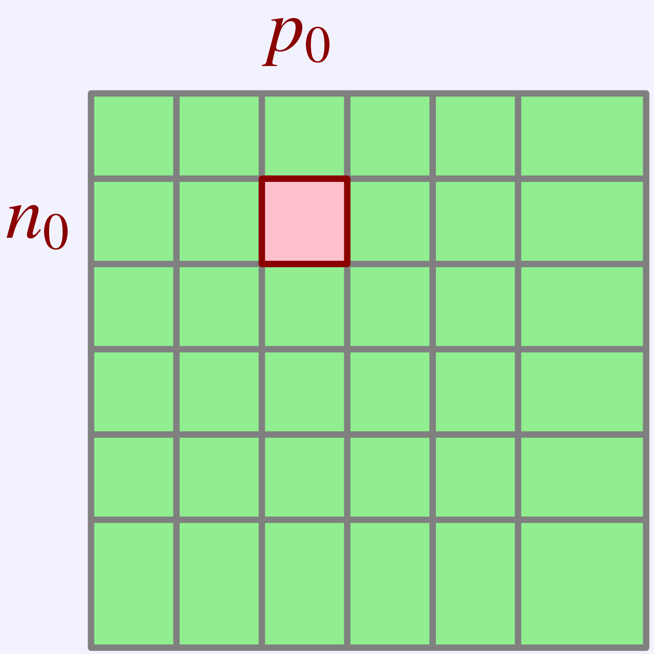
Shape of randomised algorithm

Input: $p, V, T \in \bigwedge^p V$

Output: $T \in X_p(V)$?



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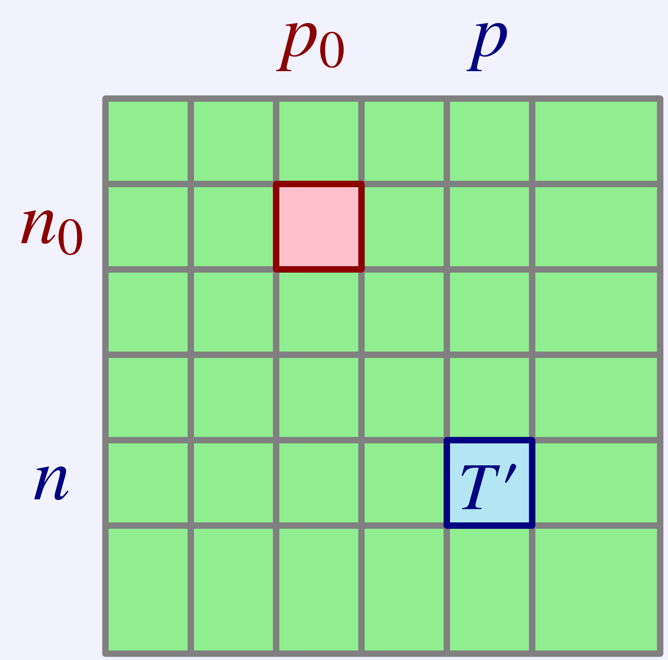

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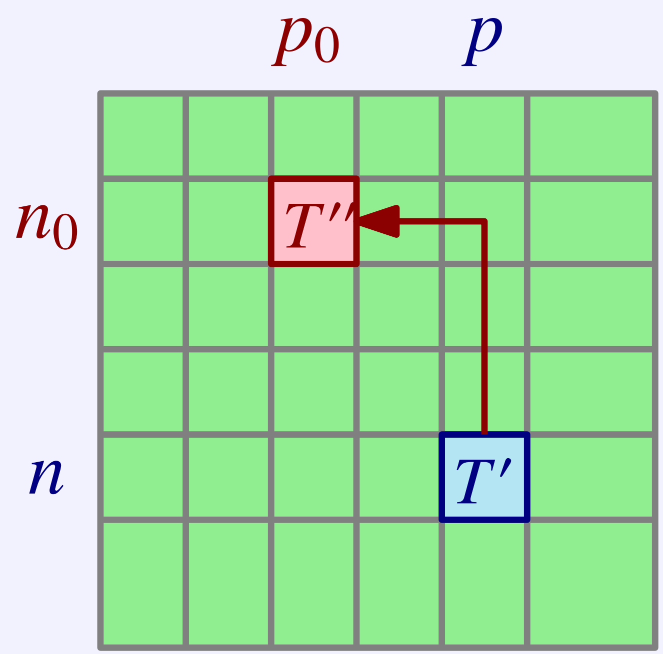
4. set $T'' :=$ image of T' in $V^*_{n_0,p_0}$
5. return $T'' \in X_{n_0,p_0}$?



A determinantal variety

$Y^{k,l} := \{T \in (\bigwedge^{\infty/2} V_\infty)^* \mid \forall g \in \mathrm{GL}_\infty :$

image of gT in $\bigwedge^2 V^*_{2k,2}$ has rank $\leq 2k$ and

image of gT in $\bigwedge^{2l} V^*_{2,2l}$ has rank $\leq 2l\}$.

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Example with $k = 2$: $\bigwedge^2 V^*_{4,2}$ has coordinates $x_{ij} = x_i \wedge x_j$, $i, j \in \{-4, -3, -2, -1, 1, 2\} \rightsquigarrow$

$$\mathrm{Pfaff}_{2k,2} = \mathrm{Pfaff} \begin{bmatrix} 0 & x_{-4,-3} & x_{-4,-2} & x_{-4,-1} & x_{-4,+1} & x_{-4,+2} \\ -x_{-4,-3} & 0 & x_{-3,-2} & x_{-3,-1} & x_{-3,+1} & x_{-3,+2} \\ \cdot & \cdot & 0 & x_{-2,-1} & x_{-2,+1} & x_{-2,+2} \\ \cdot & \cdot & \cdot & 0 & x_{-1,+1} & x_{-1,+2} \\ \cdot & \cdot & \cdot & \cdot & 0 & x_{+1,+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

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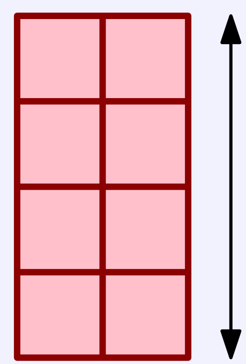

$2k$

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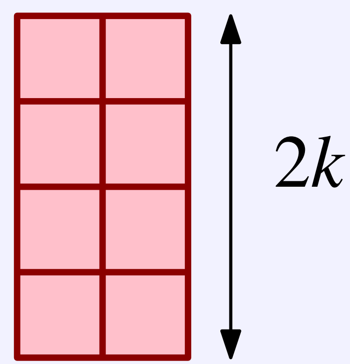
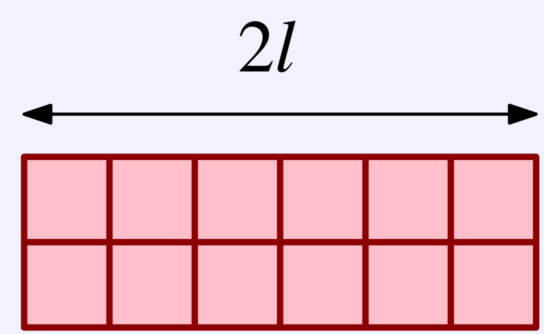


Pfaffian on $\bigwedge^{2l} V^*_{2,2l}$ has largest variable



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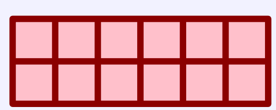
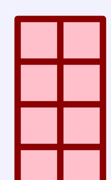
• On Z, all variables $\geq$  or $\geq$  can be expressed in smaller ones.

Proof climax

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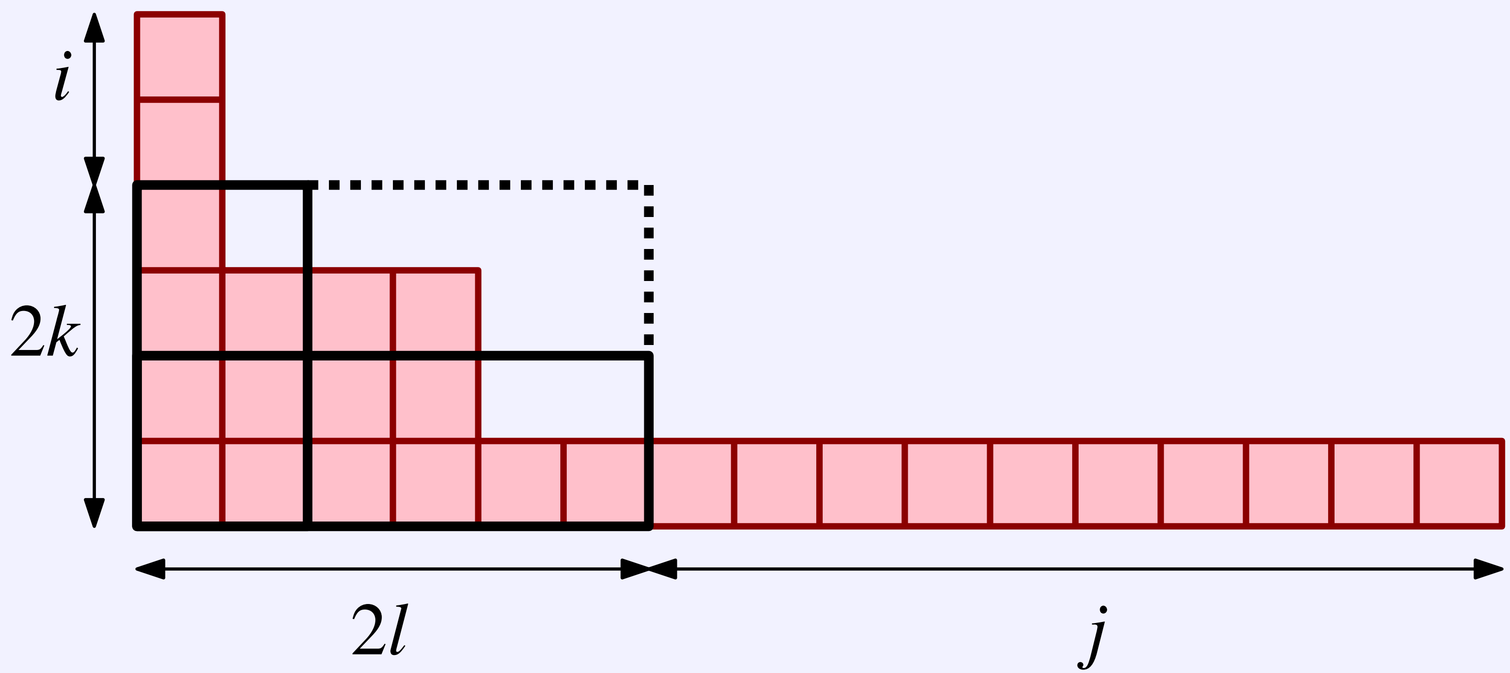

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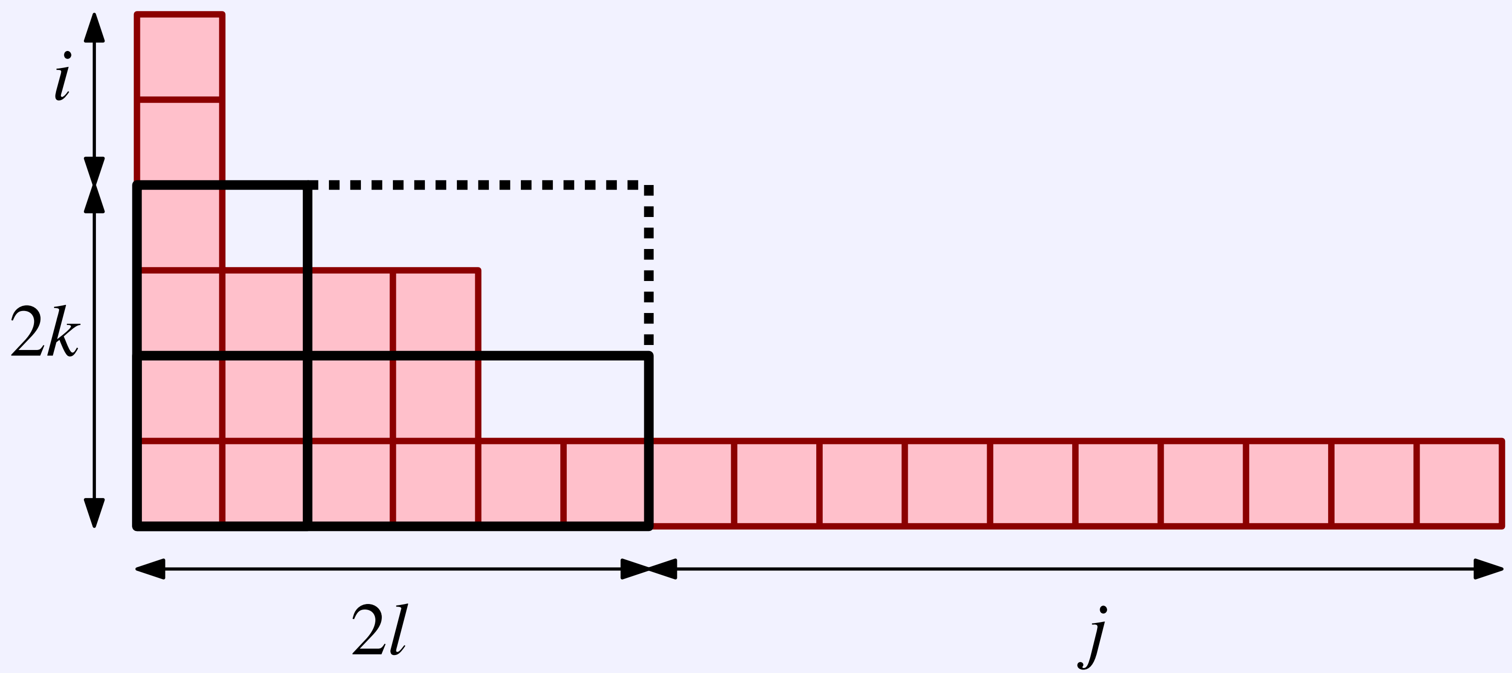

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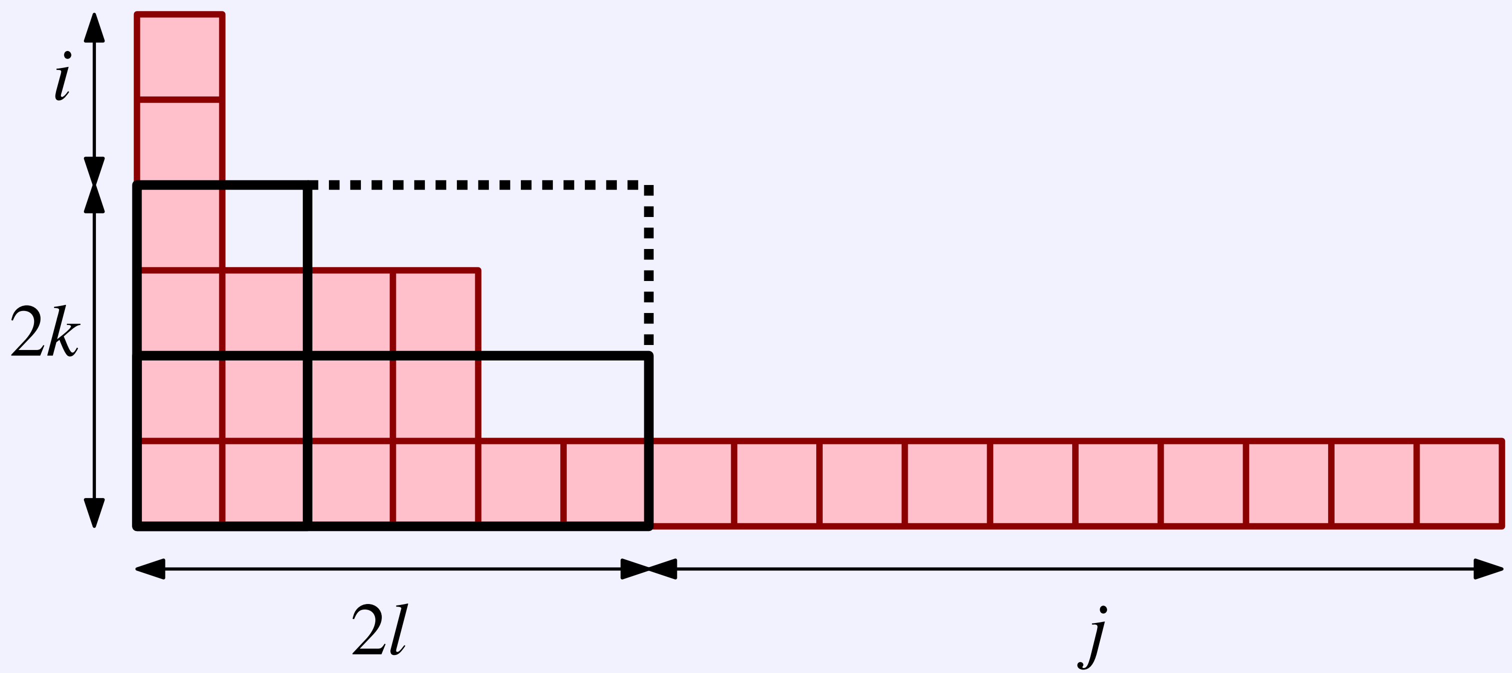

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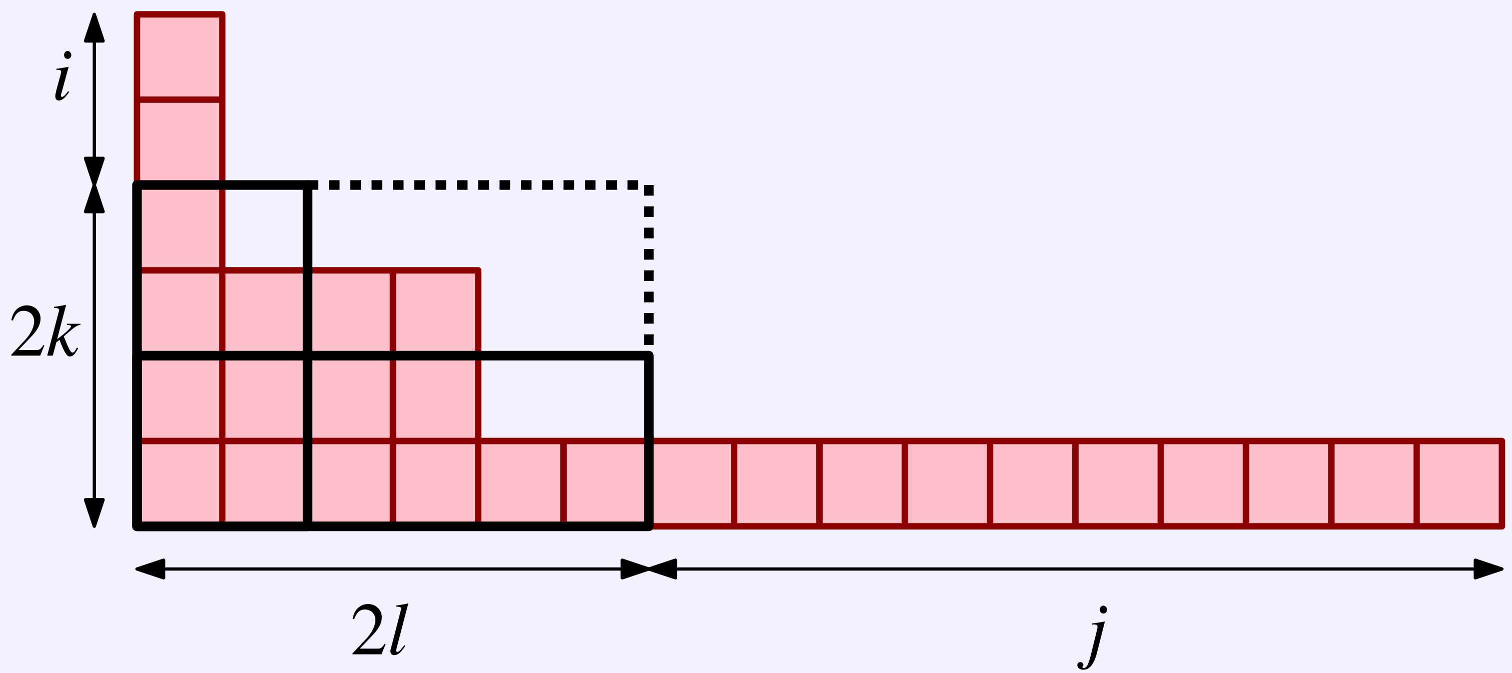

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Thank you!