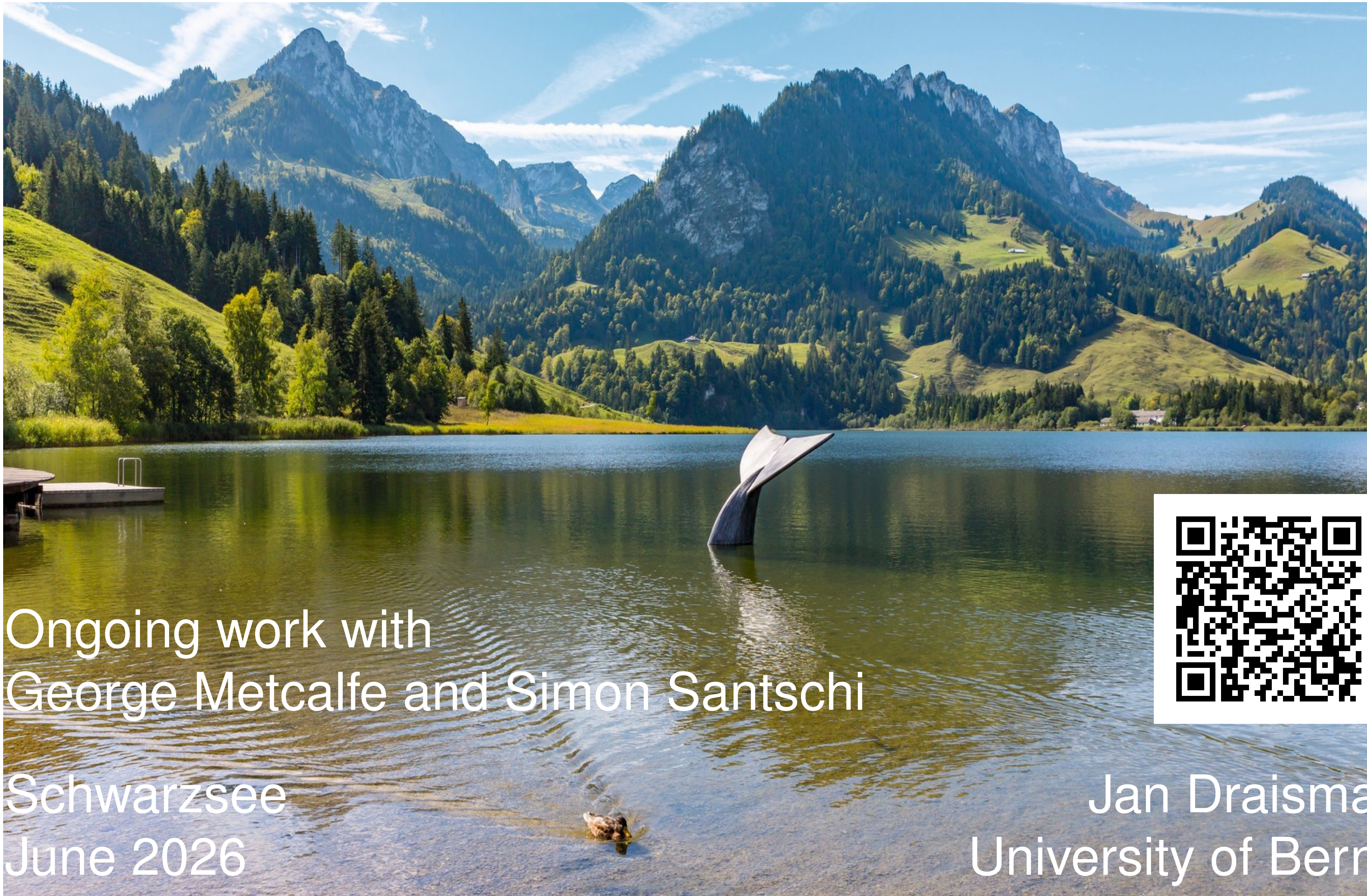
The space of monomial preorders[1]



Ongoing work with
George Metcalfe and Simon Santschi

Schwarzsee
June 2026

Jan Draisma
University of Bern

A question from universal algebra

Question [Metcalfe]

$\exists$? an algorithm that solves systems such as

$$\max\{x_1x_2x_3, x_4x_5x_6, x_7x_8x_9\} < \min\{x_1x_4x_7, x_2x_5x_8, x_3x_6x_9\}$$

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A *preorder* $\leq$ on Π:

- $(u \leq v$ and $v \leq w) \Rightarrow u \leq w$
- $u \leq u$
- $u \leq v$ or $v \leq u$
- $u \leq v \Rightarrow uw \leq vw$

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we do *not* require:

- $(u \leq v \text{ and } v \leq u) \Rightarrow u = v$
- $1 \leq u$

Examples and Robbiano's theorem

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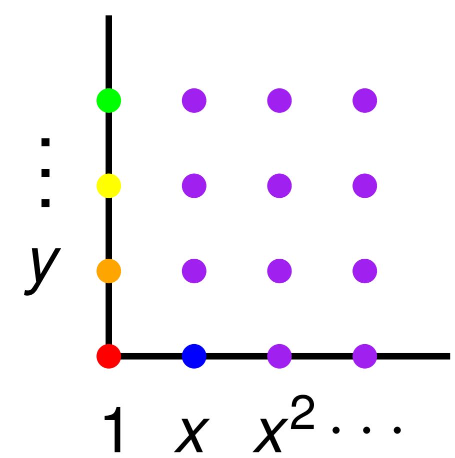

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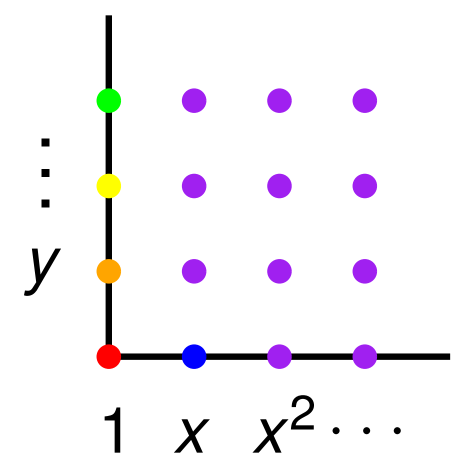

Question

Characterisation of preorders on Mon_n *à la* Robbiano?

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• In general: $\leq$ is described by $I_{\leq}$ and an *order* on monomials in $K\Pi / I_{\leq}$.

A digression

Setting

- K algebraically closed field
- $I \subseteq K[x_1, \dots, x_n]$ is called (generalised) *binomial* if it is generated by polynomials $cx^{\alpha} + dx^{\beta}$ with ≤ 2 terms.

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Original hope: use this to reduce the problem to the prime case—doesn't seem to work!

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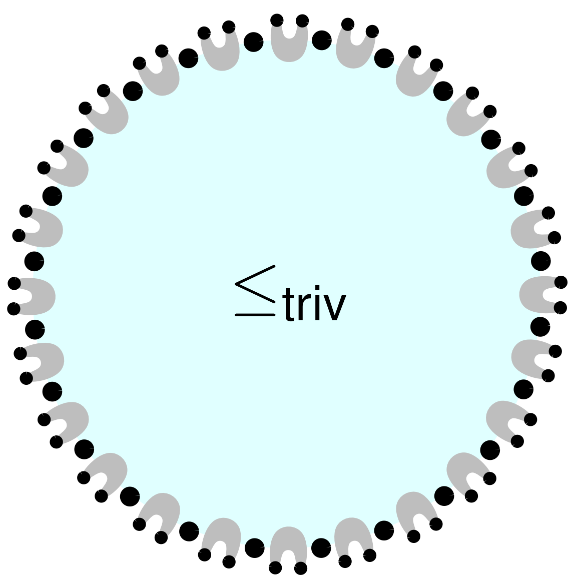

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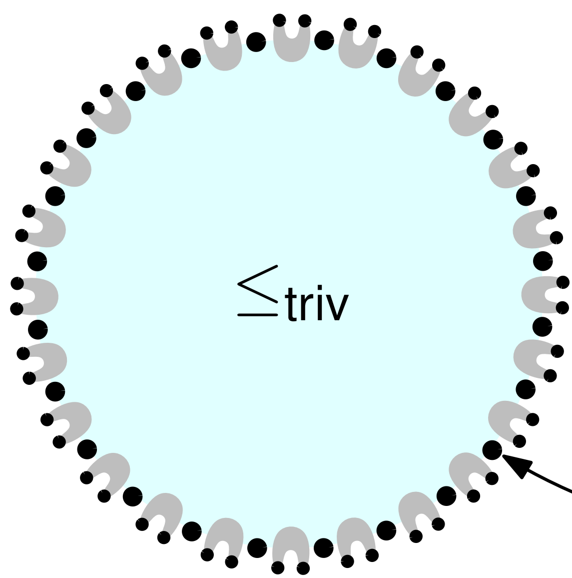

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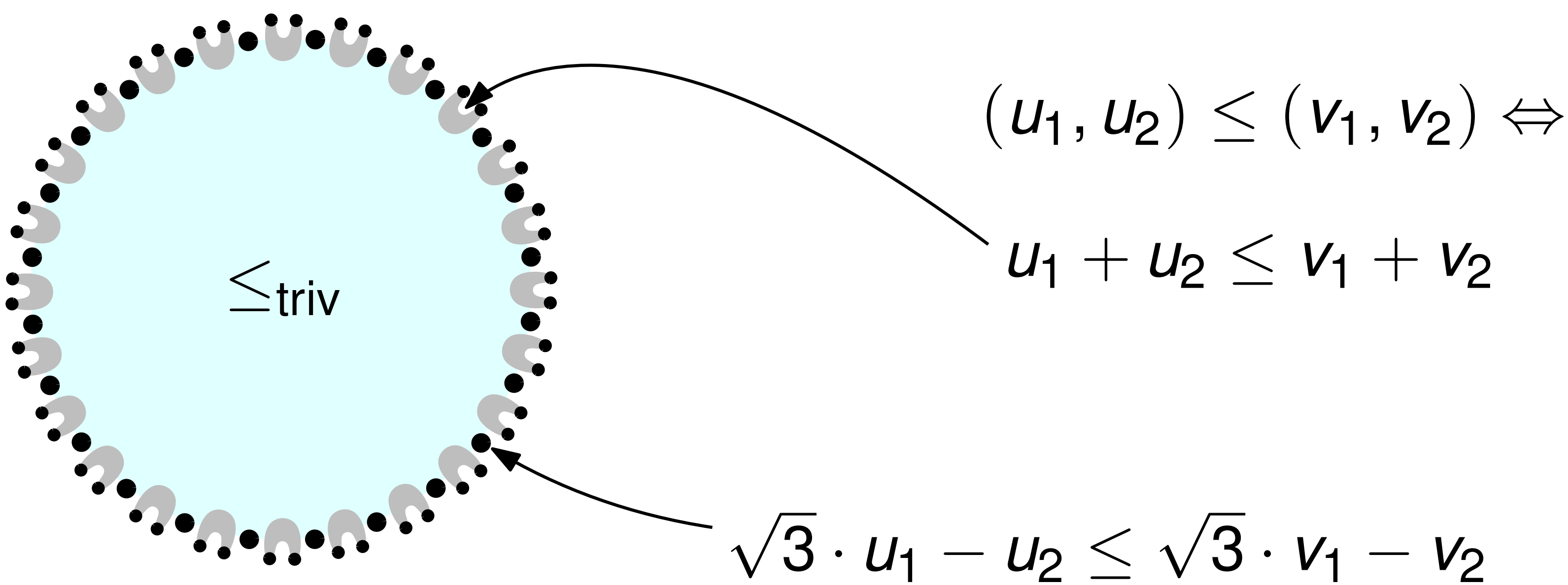

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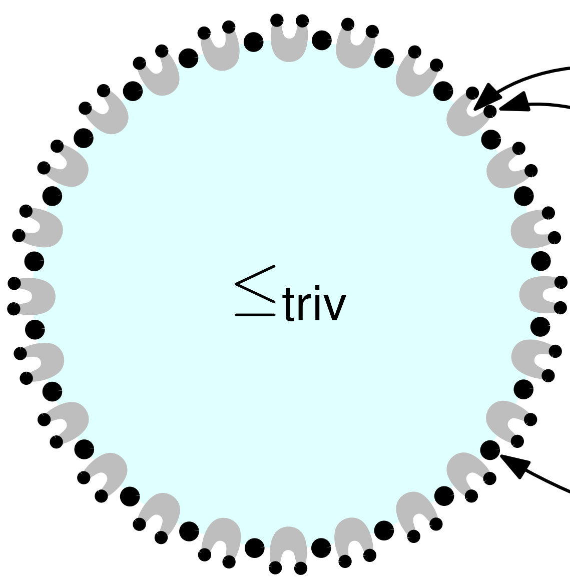

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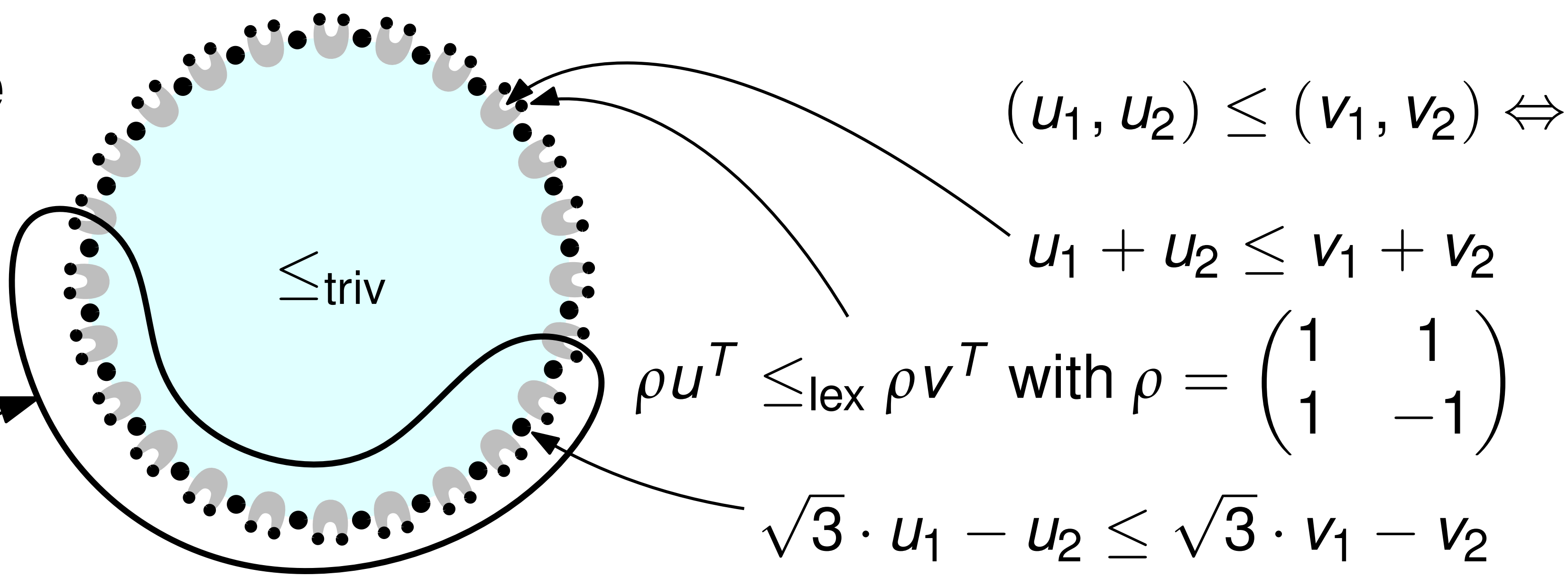

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• For any finitely generated commutative monoid Π, $\mathcal{P}(\Pi)$ is spectral, and every point is open in its closure.

• For all $n \geq 1$, $\mathcal{P}(\mathbb{Z}^n)$ has Krull dimension n but $\mathcal{P}(\mathbb{Z}^n_{\geq 0})$ has infinite Krull dimension.

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$\mathcal{P}(\mathbb{Z}_{\geq 0})$:

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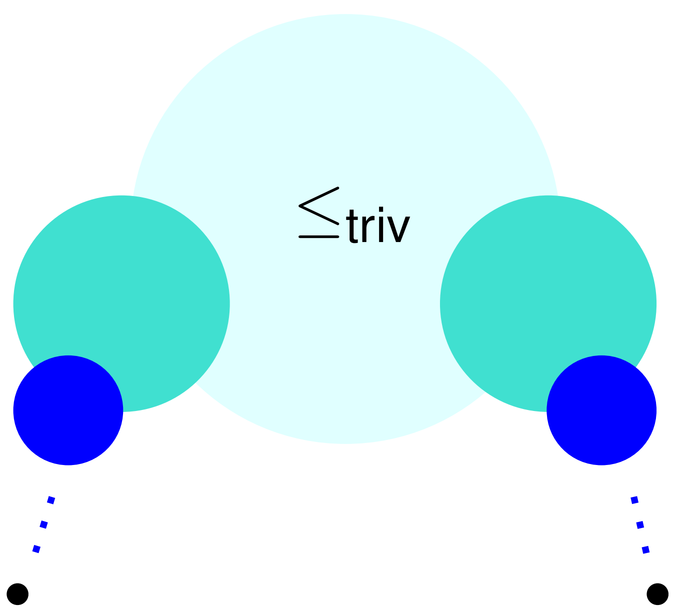

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Main results

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Theorem B: For any finitely generated commutative monoid Π, there exist admissible sets $A_1 \subseteq \mathbb{R}^{n_1}, A_2 \subseteq \mathbb{R}^{n_2}, \ldots$ and continuous maps $\varphi_i : A_i \to \mathcal{P}(\Pi)$ such that $\mathcal{P}(\Pi) = \bigcup_i \operatorname{im}(\varphi_i)$.

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Theorem D: The universal theory of (totally) ordered commutative monoids is decidable.

Definition

- A Π-*set* S is a set with a map $\Pi \times S \to S$, $(u, s) \mapsto us$ s.t. $1s = s$ and $(uv)s = u(vs)$; S is *fin. gen.* if $S = \bigcup_{i=1}^{k} \Pi s_i$.

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Rest of this talk:

1. the case where Π is a group (generalisation of Robbiano's Theorem to Π-sets); and 2. the monoid case.

1. Generalising Robbiano's theorem to Π-sets

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Proposition

$\leq$ on S nontrivial $\rightsquigarrow \exists f : \Pi \to \mathbb{R}$ homo, $g : S \to \mathbb{R}$ nonconstant s.t. $g(us) = f(u) + g(s)$ and $s \leq t \Rightarrow g(s) \leq g(t)$.

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Induction: $\Pi' := \ker(f)$, $X \subseteq \mathbb{R}$ finite set of $\operatorname{im}(f)$-coset representatives in $\operatorname{im}(g)$, $S_x := g^{-1}(x)$ $\rightsquigarrow$ $\leq$ is determined by g and $\leq|_{S_x}$ on the Π'-sets S_x for $x \in X$.

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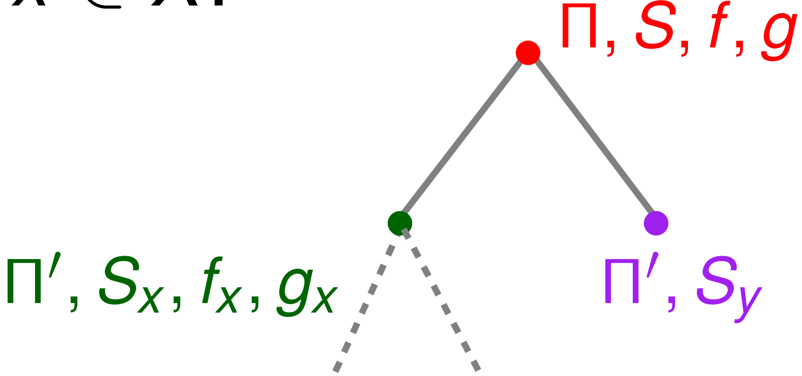

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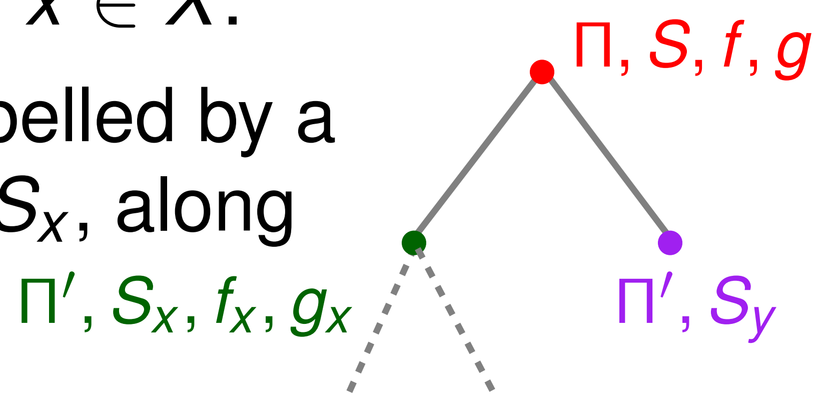

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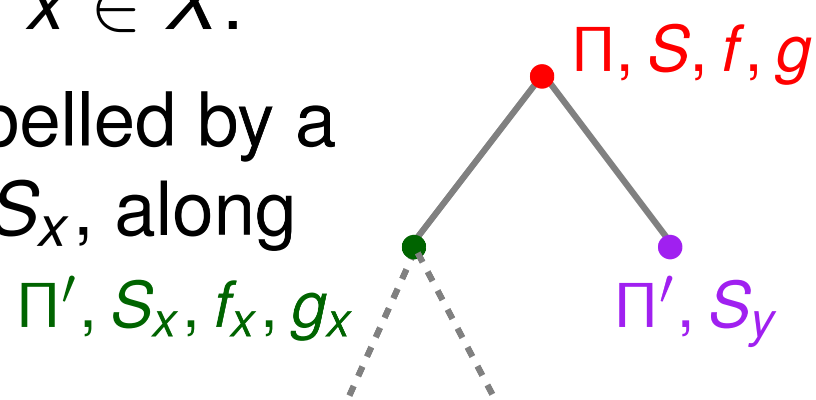
$\rightsquigarrow$ Get a *preotree* for $\leq$: vertex x labelled by a subgroup $\Pi_x \subseteq \Pi$ and a f.g. Π_x-set S_x, along with *numerical data* (f_x, g_x).



Theorem [DMS, Rust-Reid 97,...]**:** Every preorder on S comes from a preotree with numerical data, and vice versa.

2.a. Groupification

Setting: Π a f.g. commutative monoid, S a f.g. Π-set.

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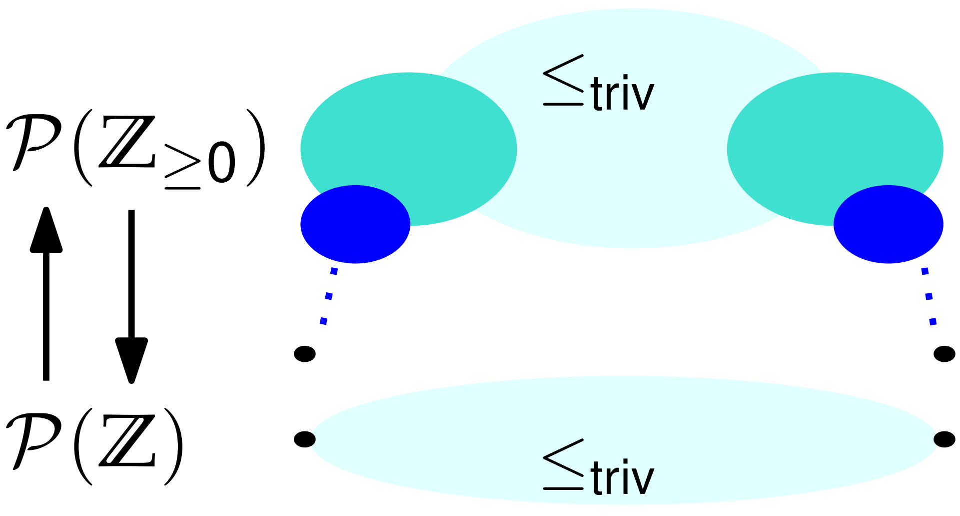
Example
$\Pi = S = \mathbb{Z}_{\geq 0}$
$\mathrm{Gr}(\Pi) = \mathrm{Gr}(S) = \mathbb{Z}$



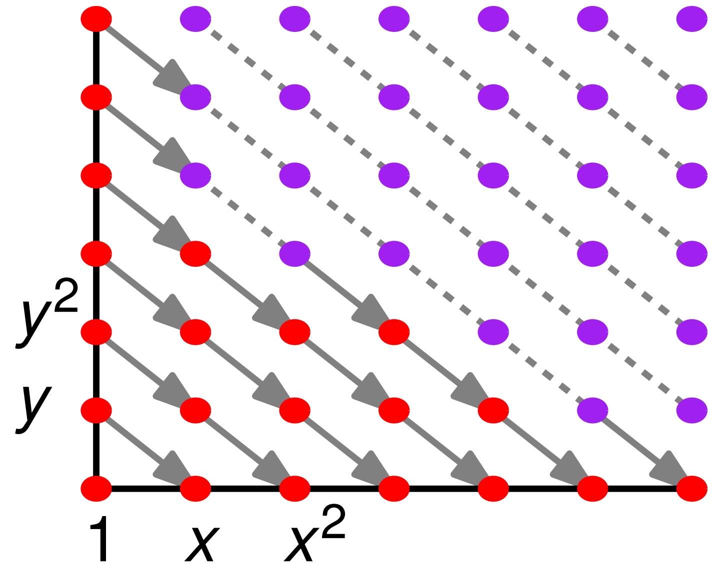
Example: $\leq$ on Mon_2

$x^i y^j \leq x^k y^l \Leftrightarrow$

$i + j < k + l$ or

$i + j = k + l$ and $i > k$ or

$i + j = k + l$ and $x^i y^j, \ldots, x^k y^l$ purple



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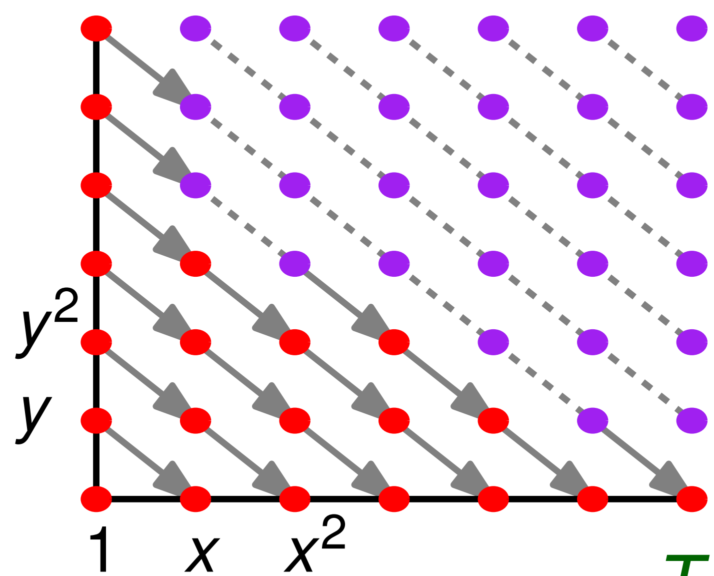


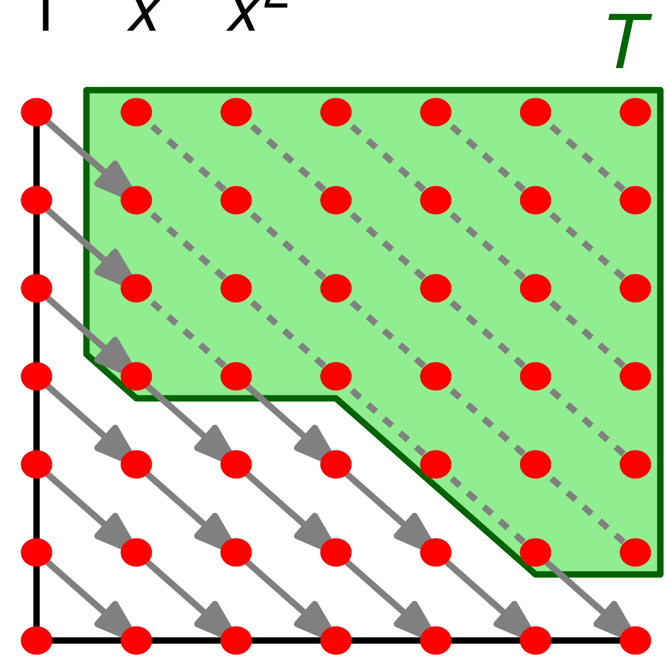
Proposition

$\leq$ on S with coarsening $\leq'$

$\rightsquigarrow$ $\exists\Pi$-stable $T \subseteq S$ s.t.

- $\forall s \in S \exists u \in \Pi : us \in T$
- T intersects each $\approx'$-class

either in $\emptyset$ or in a single $\approx$-class.



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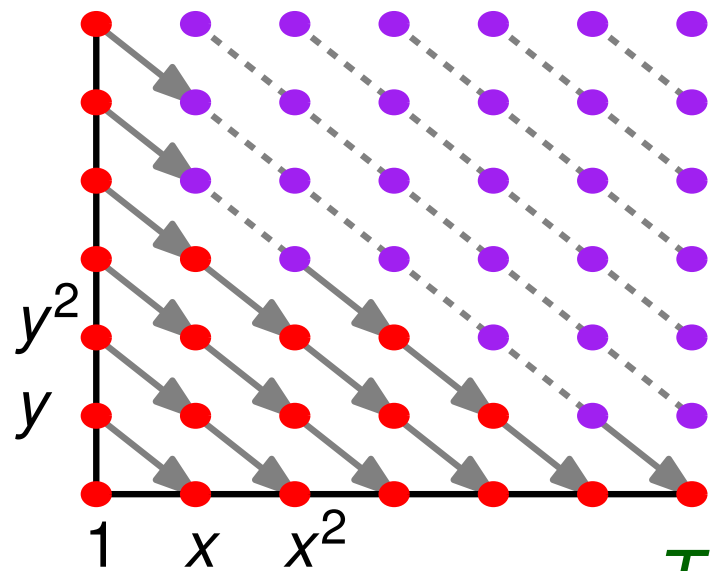

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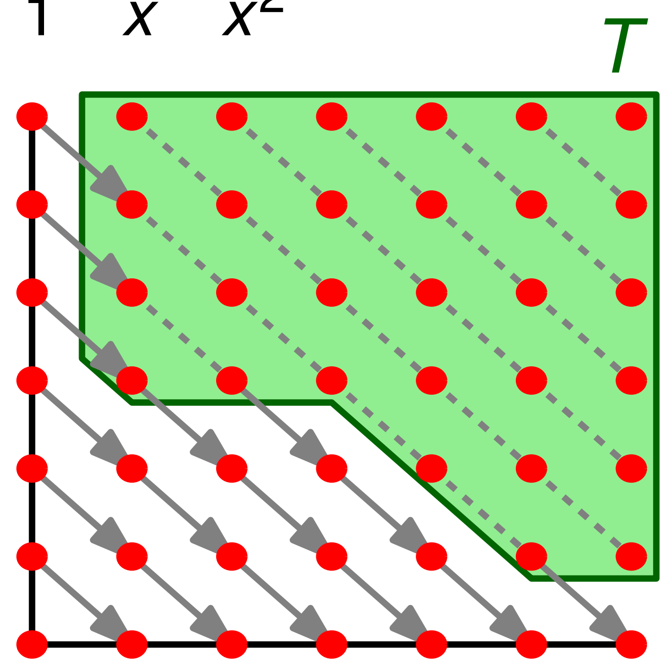
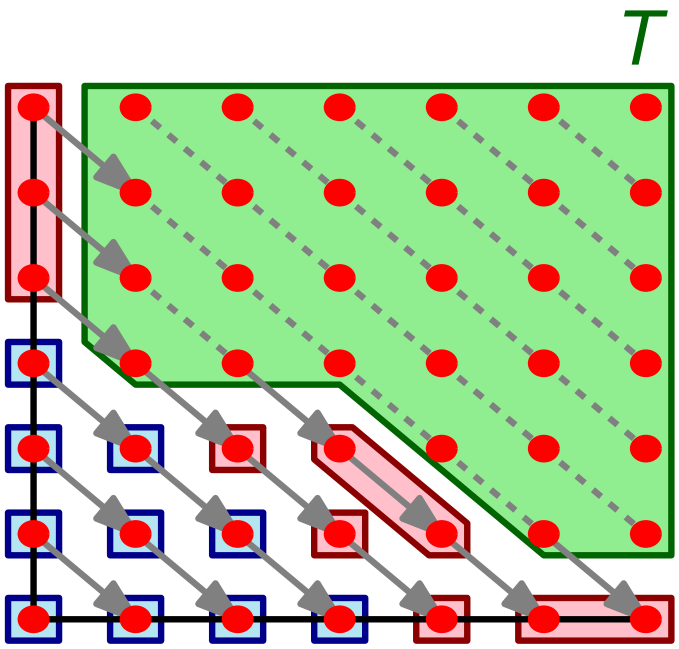


T is an *asymptotic range* for $\leq$.

Proposition: S a f.g. Π-set, $\leq$ preorder on S, coarsening $\leq' \rightsquigarrow \exists$ *finite* Π-set X, a Π-equivariant map $\varphi : S \to X$ & a partial order $\preceq$ on X, s.t.:



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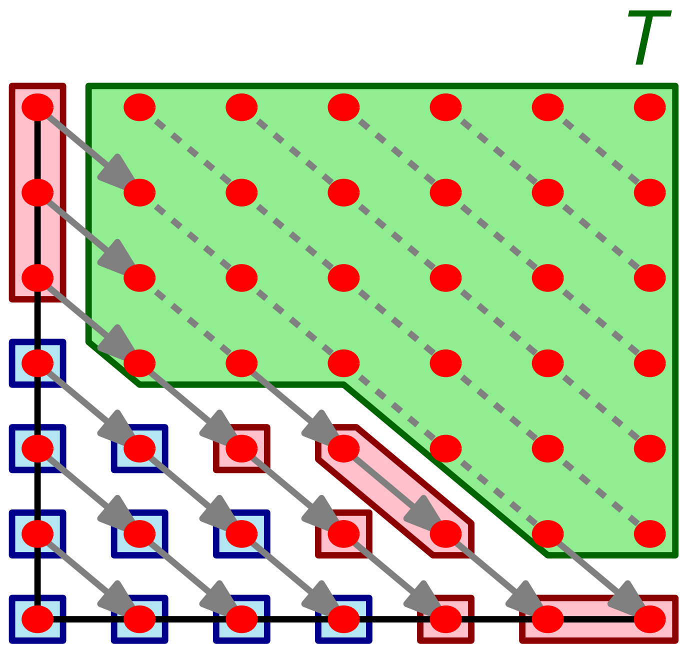

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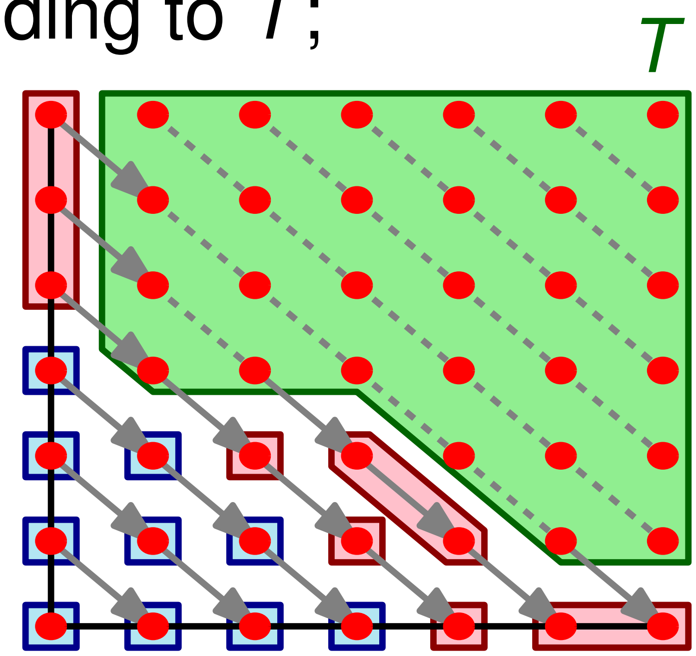

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2.c. Homomorphisms to finite Π-sets

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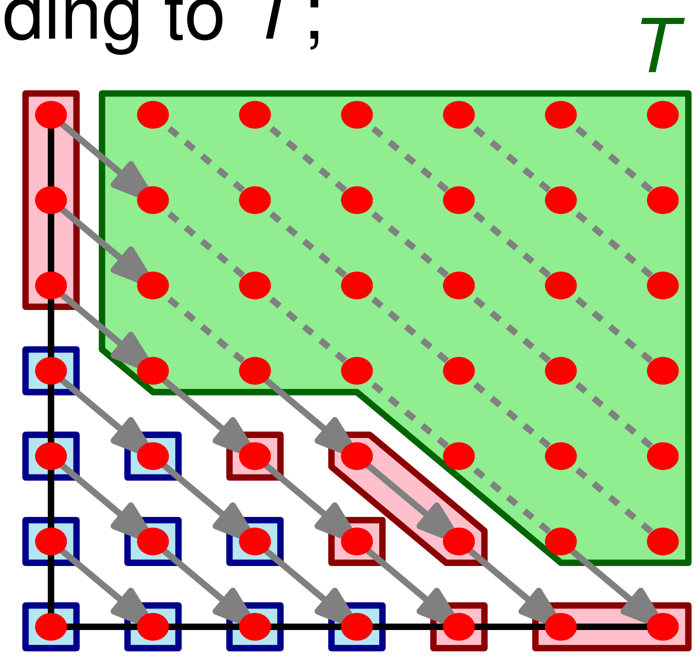

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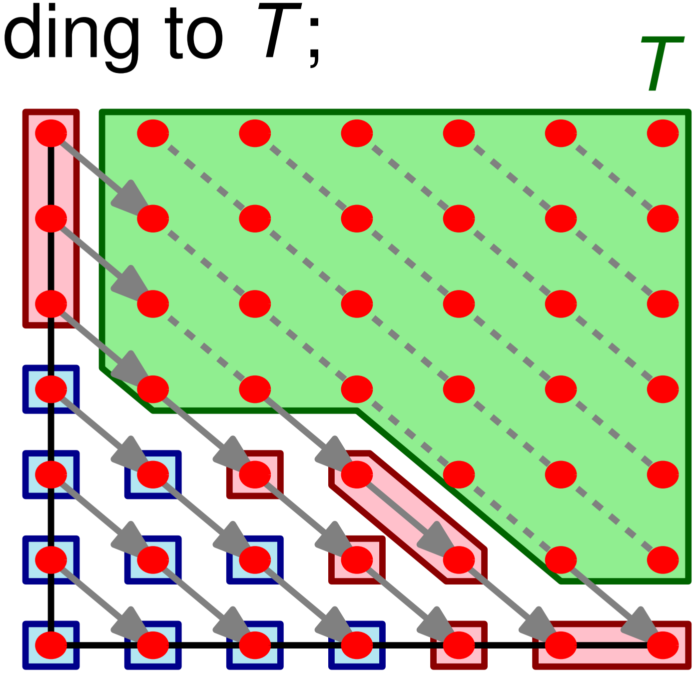
- each $S_x := \varphi^{-1}(x)$ is a f.g. Π_x-set;
- $s \leq t \Leftrightarrow$
$s <' t$ or
$s \approx' t$ and $\varphi(s) \prec \varphi(t)$ or
$s \approx' t$ and $\varphi(s) = \varphi(t) =: x$
and then $s \leq t$ in S_x.



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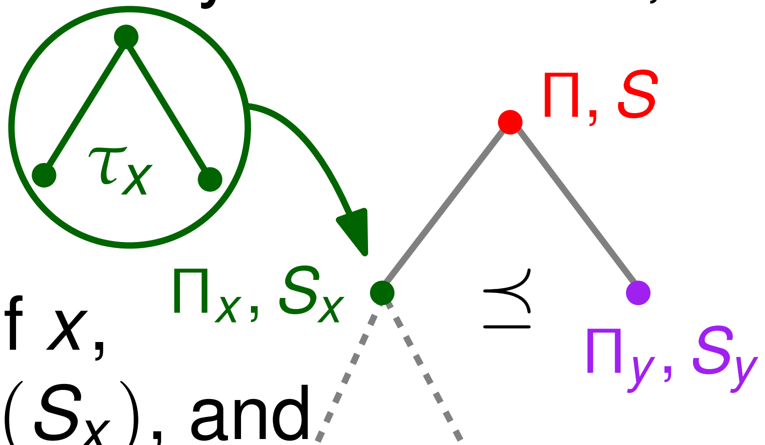
Theorem

Any preorder $\leq$ on the Π-set S is given by a finite tree, in which vertex x is labelled by:



- a f.g. submonoid $\Pi_x \subseteq \Pi$,
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Conversely, the locus of numerical data $(p_x)_x$ that gives rise to a preorder on S is a countable union of admissible sets.

2. The monoid case

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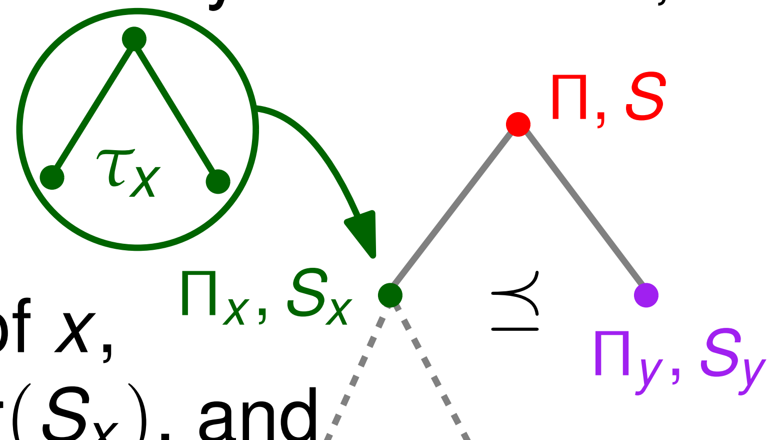

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$\rightsquigarrow$ **Theorem B**

For any finitely generated commutative monoid Π, there exist admissible sets $A_1 \subseteq \mathbb{R}^{n_1}, A_2 \subseteq \mathbb{R}^{n_2}, \ldots$ and continuous maps $\varphi_i : A_i \to \mathcal{P}(\Pi)$ such that $\mathcal{P}(\Pi) = \bigcup_i \mathrm{im}(\varphi_i)$.

Wrapping up

Question: $\exists$? a preorder on Mon_9 s.t.
$\max\{x_1x_2x_3, x_4x_5x_6, x_7x_8x_9\} < \min\{x_1x_4x_7, x_2x_5x_8, x_3x_6x_9\}$

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Theorem [Bou 15]
There is a *communication ideal* $I \subseteq \mathrm{Mon}_9$ (i.e., the set $\{(I : u) \mid u \in \mathrm{Mon}_9\}$ is totally ordered by inclusion), which contains all mons on the right but none of those on the left.

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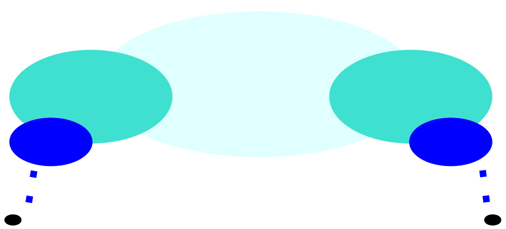

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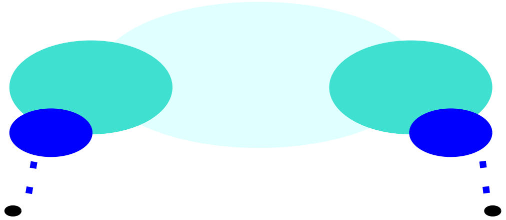

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