POLYNOMIAL FUNCTORS AND THEIR CLOSED SUBSETS: EXERCISES

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All polynomial functors are of finite degree and over an infinite field K. The following are exercises to get acquainted with the basic concepts.

(1) Let $P : \mathrm{Vec} \to \mathrm{Vec}$ be a polynomial functor. For each nonnegative integer e and each $V \in \mathrm{Vec}$ define

$$P_e(V) := \{q \in P(V) \mid \forall t \in K : P(t1_V)q = t^e q\}.$$

   (a) Show that P_e is a polynomial subfunctor of P.
   (b) Show that $P \cong \bigoplus_e P_e$.

(2) Let P be the polynomial functor over $K = \mathbb{C}$ that sends V to the space of symmetric tensors in $V \otimes V$. So $P(\mathbb{C}^n)$ is the space of complex symmetric $n \times n$-matrices with GL_n-action given by $(g, A) \mapsto gAg^T$.
   (a) Show that for a fixed n, GL_n has only finitely many orbits on $P(\mathbb{C}^n)$, parameterised by rank.
   (b) Show that the Vec-closed subsets of P form a single chain $X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_\infty = P$.

(3) Let P be the same polynomial functor as in the previous exercise. For a pair $(A, B) \in P(V) \oplus P(V)$ define the *tuple rank* $\mathrm{rk}(A, B)$ as the minimal rank of $sA + tB$ over all pairs $(s, t) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Show that for each k, X_k defined by $X_k(V) = \{(A, B) \in P(V) \oplus P(V) \mid \mathrm{rk}(A, B) \le k\}$ is a Vec-closed subset of $P \oplus P$.

(4) Recall that $P_\infty = \lim_{\leftarrow n} P(K^n)$, and for X a Vec-closed subset of P we have $X_\infty = \lim_{\leftarrow n} X(K^n)$. Show that $X \mapsto X_\infty$ is a bijection between Vec-closed subsets of P and GL_∞-stable closed subsets of P_∞.

(5) For a fixed $U \in \mathrm{Vec}$ let $\mathrm{Sh}_U : \mathrm{Vec} \to \mathrm{Vec}$ be the functor that sends V to $U \oplus V$ and $\varphi : V \to W$ to $1_U \oplus \varphi$. For any polynomial functor P of degree $\le d$ show that $Q = P \circ \mathrm{Sh}_U$ is a polynomial functor of degree $\le d$ and that the top-degree parts are isomorphic: $P_d \cong Q_d$.

(6) Formulate a Vec^k-version of the main theorem and prove it using the main theorem.

(7) Prove or give a counterexample: if P is a polynomial functor over K with $P_0 = 0$, then P has precisely two Vec-*open* subsets X (i.e., $X(V)$ open in $P(V)$ and for all $\varphi \in \mathrm{Hom}(V, W), P(\varphi)X(V) \subseteq X(W)$). In particular, the complement $Y^c(V) : P(V) \setminus Y(V)$ of a Vec-closed subset Y is typically not Vec-open. Show, however, that the complement Y^c is preserved under *injective* linear maps.

The following are some open problems, in increasing difficulty (I think).

(1) (Erman-Sam-Snowden) Formulate and prove a $\mathbb{Z}$-version of the main theorem.

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(2) Is the ideal of the variety of tensors of the form $v_1 \otimes A_{23} + v_2 \otimes A_{13}$ in $V^{\otimes 3}$ (with v_i in the i-th copy of V and A_{ij} in the tensor product of the i-th and j-th copies) generated by polynomials of degrees 3 and 6? (I can tell you some of these equations.)

(3) Does every increasing chain of GL_∞-stable ideals in $\mathbb{C}[x_{ij} | i, j \in \mathbb{N}]$, with action $(g, A) \mapsto gAg^T$ stabilise? This is a special case of the ring-theoretic Noetherianity we'd like to have.