POLARISATION IN INVARIANT THEORY

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Invariant Theory

Set-up:

- V a finite-dimensional vector space over a field K
- G a group acting by linear maps on V
- $K[V]$ the ring of polynomials on V
- G acts on $K[V]$ by $gf := f \circ g^{-1}$
- $K[V]^G$ the algebra of (polynomial) G-invariants on V

Example 0.1. $V := K^n$, $G := \{g \in \mathrm{GL}_n \mid g^T g = I\}$; then $K[V]^G = K[x_1, \ldots, x_n]^G = K[q]$, where $q = x_1^2 + \ldots + x_n^2$.

Invariants help to distinguish G-orbits in V:

- If $x, y \in V$ and $f \in K[V]^G$ satisfies $f(x) \neq f(y)$, then Gx and Gy are distinct orbits.
- For finite groups, distinct orbits are seperated by invariant polynomials, but this is not true for general groups.

Two problems in invariant theory

Generators. Given G and its action on V, determine generators of $K[V]^G$.

Remark 0.2.

- $K[V]^G$ may not be a finitely generated algebra; this is the negative answer to Hilbert's 14th problem (Nagata 1959).
- If the image of G in $\mathrm{GL}(V)$ is "reductive", then $K[V]^G$ is finitely generated (Hilbert, 1890), and one can give good upper bounds on the degrees of generators (Derksen, 2001).

Separating invariants. Given G and its action on V, determine a finite subset of $K[V]^G$ with the same "separating power" as the full algebra $K[V]^G$. Such a set is called a *finite separating system of invariants*.

Lemma 0.3. *A finite separating system of invariants always exists.*

Proof. Consider the equivalence relation on V given by

$$\{(v, w) \mid f(v) = f(w) \text{ for all } f \in K[V]^G\}.$$

This is the zero set of the ideal I generated by the polynomials $f(v) - f(w)$, where f runs over all elements of $K[V]^G$. By the fact that $K[V \times V]$ is Noetherian (i.e., every ascending chain of ideals stabilises), I is already generated by a finite number of the $f(v) - f(w)$. The corresponding f form a finite separating system. □

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Note that a finite set of invariants has the same separating power as the algebra it generates. So we may also interpret the lemma as: there always exists a finitely generated algebra of invariants that has the same separating power as the full algebra $K[V]^G$.

Polarisation

Now suppose that $V = M^p$, where

- M is a finite-dimensional vector space on which G acts linearly, and
- G acts diagonally on M^p.

In general one cannot construct all invariants on M^p from invariants on M.

Example 0.4. Suppose that $G = \mathrm{SL}_n$ acts on $M = K^n$ by multiplication. Then $K[M^p]^G = K$ for $p < \dim M$, while $K[M^{\dim M}] = K[\det]$.

However, one can always construct *some* invariants on M^p from knowledge of invariants on M^q, as follows. Suppose that $f \in K[M^q]^G$ and let π be any linear map $K^p \to K^q$. Then π induces a linear map $\pi : M^p \to M^q$ by

$$\pi(m_1, \ldots, m_p) = (\sum_{j=1}^{p} \pi_{1,j} m_j, \ldots, \sum_{j=1}^{p} \pi_{q,j} m_j).$$

Clearly, this map is G-equivariant: $\pi(gv) = g\pi(v)$ for $v \in M^p$. Hence $f \circ \pi$ is an invariant on M^p, which we will call a *polarisation* of f to M^p.

Definition 0.5. Let A be a subalgebra of $K[M^q]$. The subalgebra of $K[M^p]$ generated by all polarisations of elements of A to M^p (i.e., by all functions of the form $f \circ \pi$ where $f \in A$ and $\pi : K^p \to K^q$) is called the *polarisation* of A to M^p.

Example 0.6. Recall Example 0.1. Then the polynomial β on $(K^n)^2$ given by

$$\beta(x, y) := q(x + y) - q(x) - q(y)$$

is a polarisation of q to 2 copies of K^n: the bilinear form associated to q.

Applications of polarisation

Generating invariants. The following theorem is due to Weyl.

Theorem 0.7 (Weyl, 1939)**.** *Suppose that* $\mathrm{char}\, K = 0$. *Then* $K[M^p]^G$ *is the polarisation of* $K[M^{\dim M}]^G$ *to* M^p *for all* $p \geq \dim M$.

In other words, one needs "only" know the invariants of G on $\dim M$ copies of M to construct the invariants of G on more copies. Compare this to example 0.4.

The theorem is not true in positive characteristic, not even if G is finite and its order is not a multiple of $\mathrm{char}\, K$.

Example 0.8. (Kemper, Wehlau). Suppose that K has characteristic 3 and contains a primitive 4-th root ω of unity. Let G be the subgroup of GL_1 generated by ω, acting on $M = K$ by multiplication. Then $K[M]^G = K[x]^G = K[x^4]$. Now x^2y^2 is an invariant on M^2, but in $(ax + by)^4$ the monomial x^2y^2 does not occur; this easily implies that x^2y^2 does not lie in the polarisation of $K[x^4]$ to 2 copies.

Remark 0.9. Friedrich Knop has proved a generalisation of Weyl's theorem to positive characteristic, which ensures that the invariants on M^p *up to a certain degree* can be obtained from invariants on $M^{\dim M}$ by polarisation. Combining this with a bound on degrees of invariants of *finite* groups, one finds a generalisation of Weyl's theorem to finite G and $\operatorname{char} K$ not dividing $|G|$.

Separating invariants. In contrast, we have

Theorem 0.10 (Kemper, Wehlau, Draisma, 2005)**.** *If A is a separating subalgebra of $K[M^{\dim M}]^G$, then the polarisation of A to M^p with $p \geq \dim M$ is separating.*

This theorem is a consequence of the following lemma (by taking "having the same value under all invariants" as equivalence relations).

Lemma 0.11. *Suppose $p, q \geq \dim M$ and let $\sim$ and $\equiv$ be equivalence relations on M^p and M^q, respectively, such that*

$$v \sim w \Rightarrow \pi v \equiv \pi w, \quad v, w \in M^p, \pi : K^p \to K^q$$

and

$$v \equiv w \Rightarrow \pi v \sim \pi w, \quad v, w \in M^q, \pi : K^q \to K^p.$$

Now suppose that $v, w \in M^p$ are such that $\pi v \equiv \pi w$ for all $\pi : K^p \to K^q$. Then $v \sim w$.

Proof. View M^p and M^q as $K^p \otimes M$ and $K^q \otimes M$. Choose linearly independent subspaces A, B, C of K^p with

(1) $v \in (A + B) \otimes M$ (write $v = v_A + v_B$ accordingly),
(2) $w \in (B + C) \otimes M$ (write $w = w_B + w_C$ accordingly) and
(3) $A + B$ and $B + C$ have dimension at most q.

Now let $\pi : K^p \to K^q$ and $\sigma : K^q \to K^p$ be such that $\sigma\pi$ is the identity on $A + B$ and zero on C. Then we find

$$v = \sigma\pi v \sim \sigma\pi w = w_B.$$

Similarly, using a second pair (σ, π), we find $w \sim v_B$. But now let $\pi : K^p \to K^q$ and $\sigma : K^q \to K^p$ be a third pair such that $\sigma\pi$ is the identity on B and zero on $A + C$. Then we find

$$v_B = \sigma\pi v \sim \sigma\pi w = w_B,$$

and we are done. □

The null-cone. The *null-cone* $N(M^p)$ in M^p is the set of elements of V that cannot be separated from 0 by invariants.

Example 0.12. Suppose that $\mathrm{SL}_n \times \mathrm{SL}_n$ acts by left-and-right multiplication on $M = M_n$.

- The null-cone $N(M)$ consists of the singular matrices: these are the ones that cannot be distinguished from 0 by the det.
- The null-cone $N(M^p)$ for $p > 1$ has precisely n irreducible components, namely:

$$C_k := \{(A_1, \ldots, A_p) \in M^p \mid \exists k\text{-dimensional } U : \dim \sum_i A_i U < k\}.$$

Theorem 0.13 (Bürgin, Draisma, 2005)**.** *The function $p \mapsto$ "the number of irreducible components of $N(M^p)$" is ascending and stabilises at some $p \leq \dim M$.*

Remark 0.14. For reductive groups in characteristic zero, this was first observed by Kraft and Wallach (2004).

Proof. The stabilising part is the hardest. Set $q := \dim M$. Consider the map

$$\Psi : \mathrm{Hom}(K^q, K^p) \times M^q \to M^p,\ (\pi, v) := \pi v.$$

Verify:

(1) Ψ maps $\mathrm{Hom}(K^q, K^p) \times N(M^q)$ into $N(M^p)$,
(2) Ψ maps $\mathrm{Hom}(K^q, K^p) \times N(M^q)$ onto $N(M^p)$ (here we need $q = \dim M$), and
(3) the number of irreducible components of $\mathrm{Hom}(K^q, K^p) \times N(M^q)$ equals the number of irreducible components of $N(M^q)$.

□