Bounded-rank tensors and group-based models

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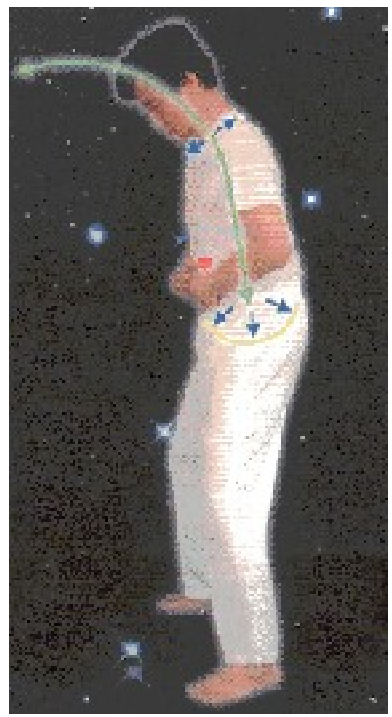
TU Eindhoven

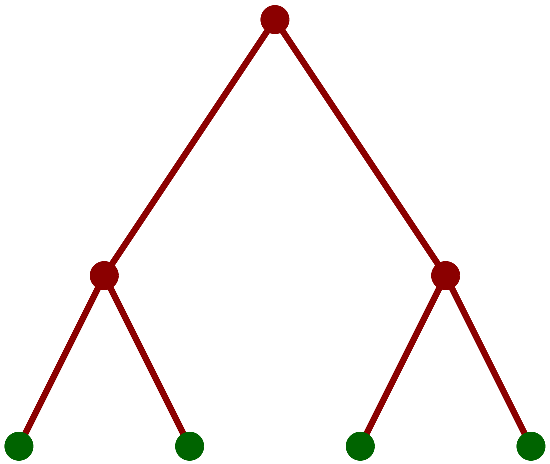
(with Rob Eggermont, Jochen Kuttler)

Algebraic Statistics, Penn State, June 2012

Flatten and contract!



Tree models



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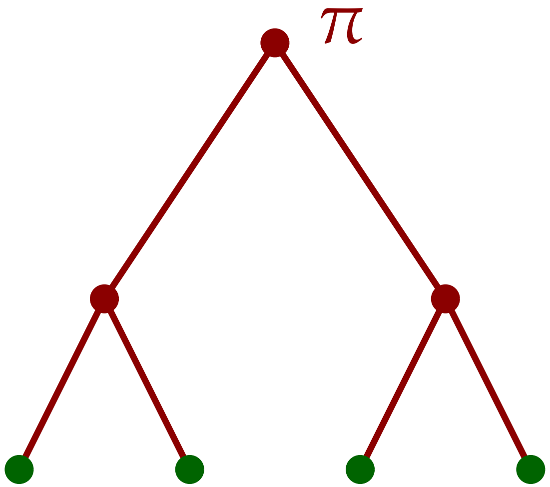

B alphabet

π distribution on B

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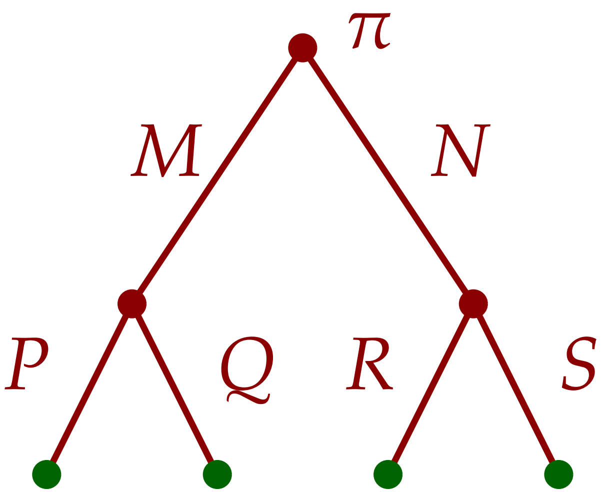

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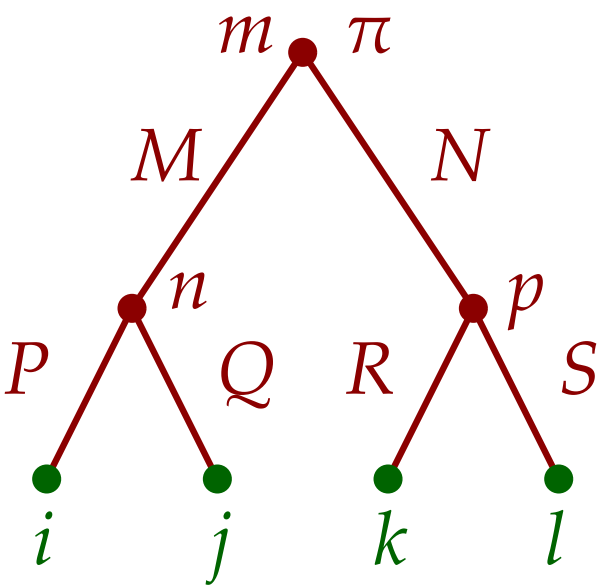

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$\text{Prob}(i, j, k, l) = \sum_{m,n,p} \pi_m M_{mn} P_{ni} Q_{nj} N_{mp} R_{pk} S_{pl}$

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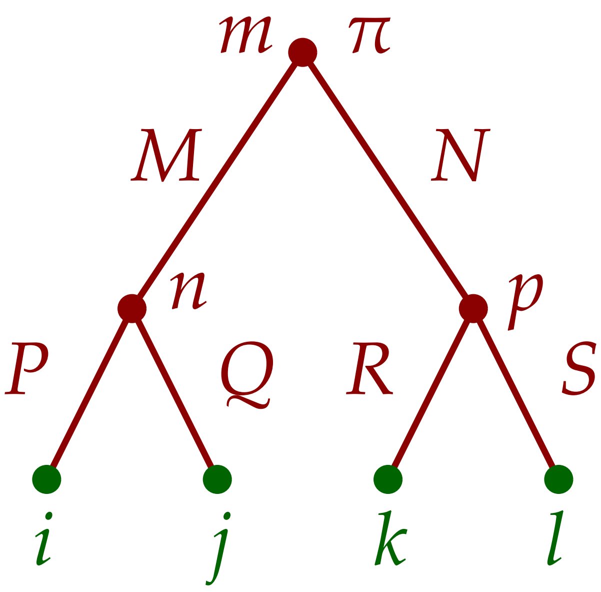

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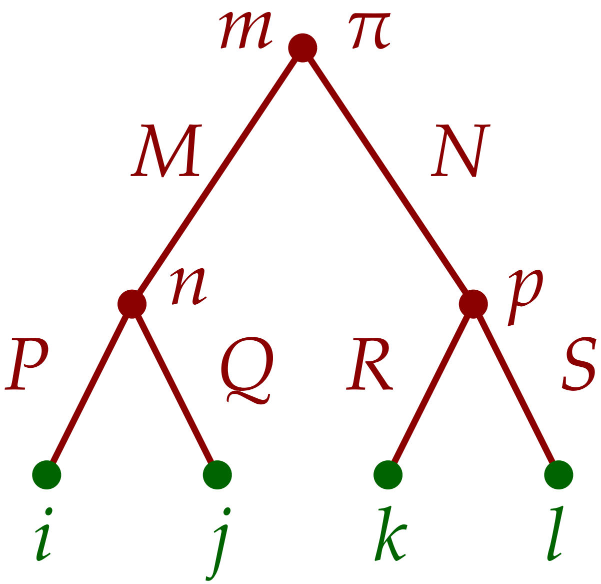

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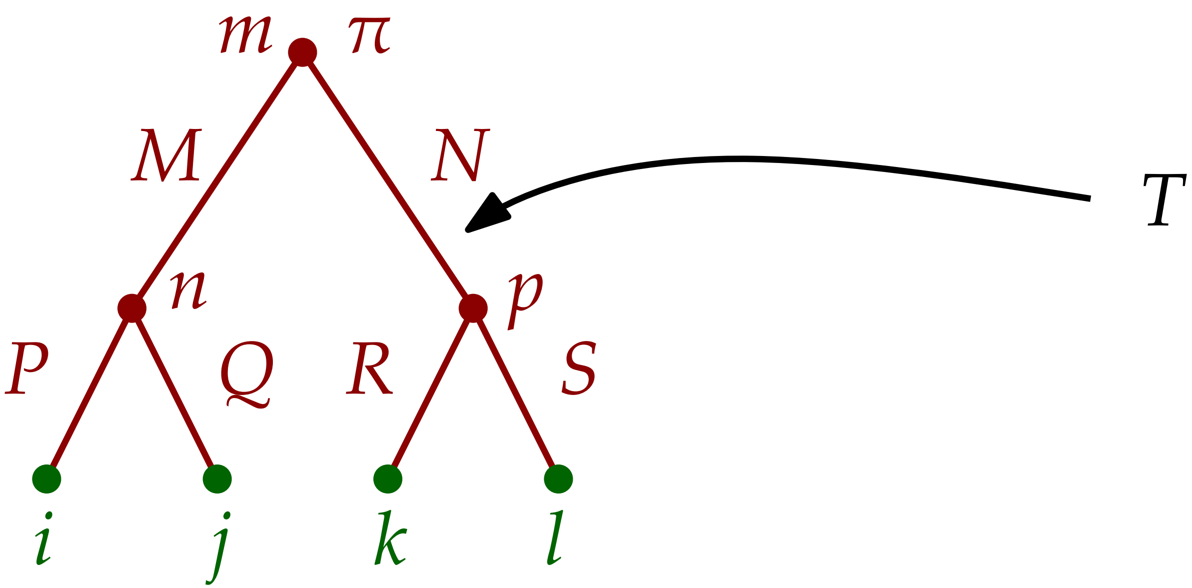

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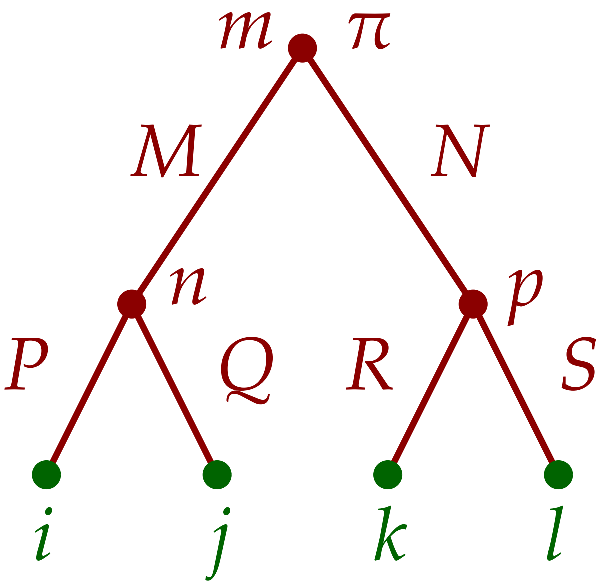
Goal: decide membership of GM(T)

Equivariant tree models



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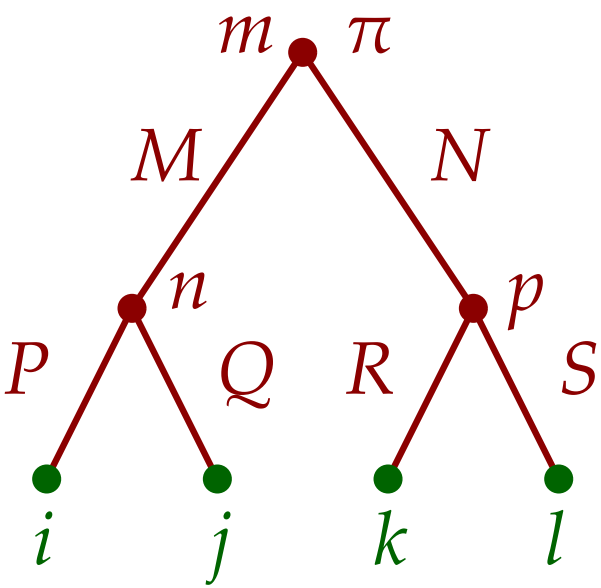

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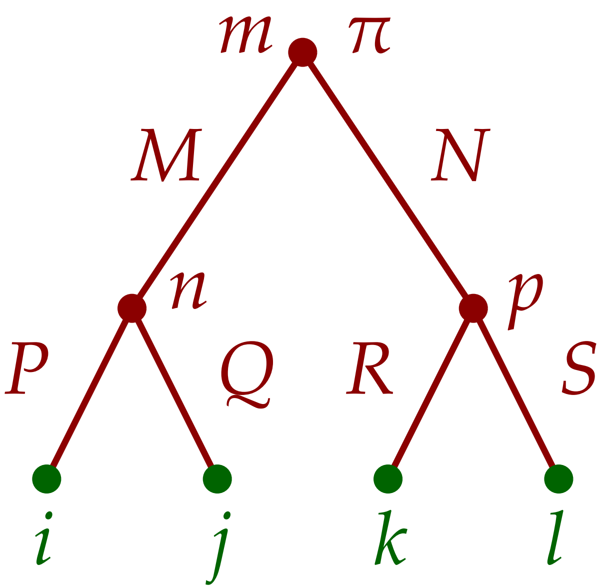

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Goal: decide membership of $\mathrm{EM}(T)$

Equivariant tree models, examples

General Markov (Allman-Rhodes, Friedland-Gross, Bates-Oeding, . . . trivalent trees)

$G = \{1\}$, $\mathrm{EM}(T) = \mathrm{GM}(T)$

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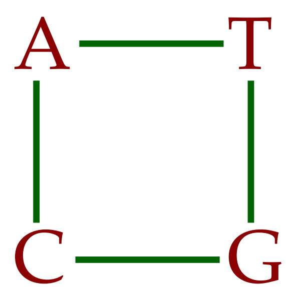

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Disclaimer

- What's the bound? What's the algorithm?
- Polynomial in $|V|^{|\mathrm{leaf}(T)|}$ can still be very slow.
- No ideal-theoretic result.

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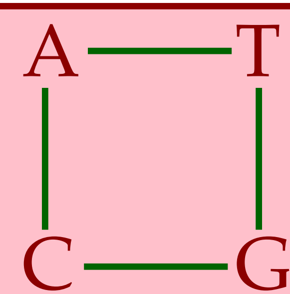
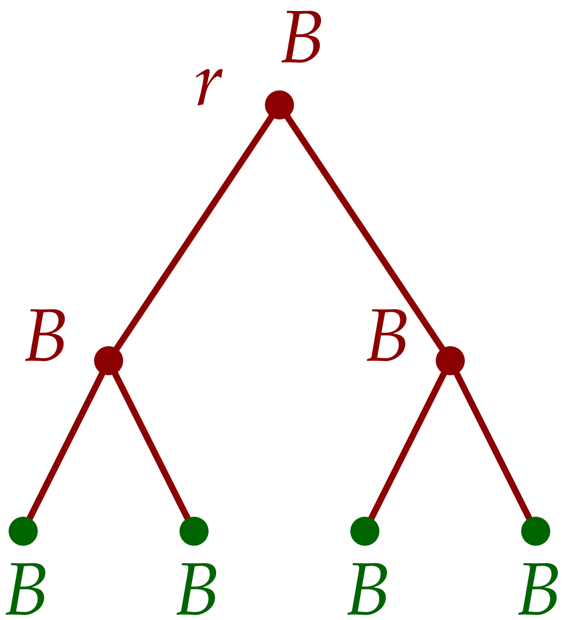


not abelian

Flatten and contract!

Flattening (I)



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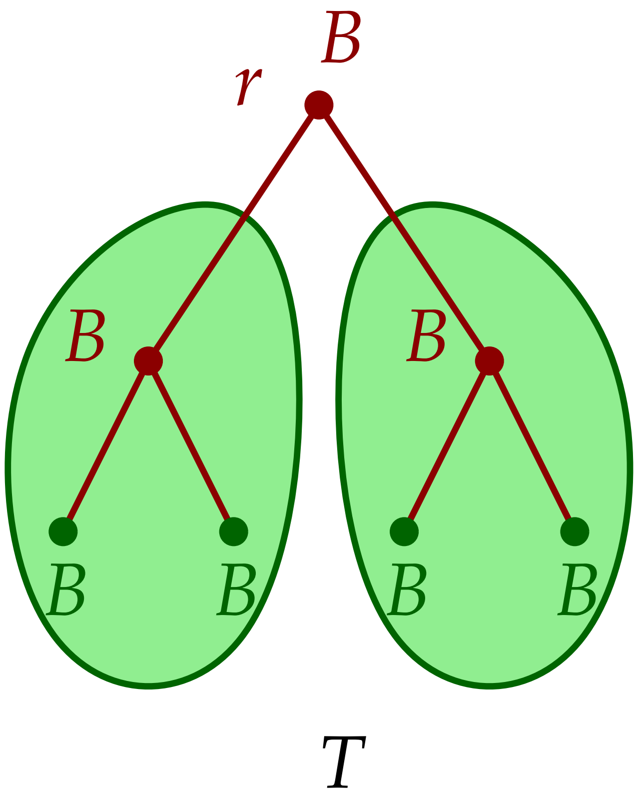

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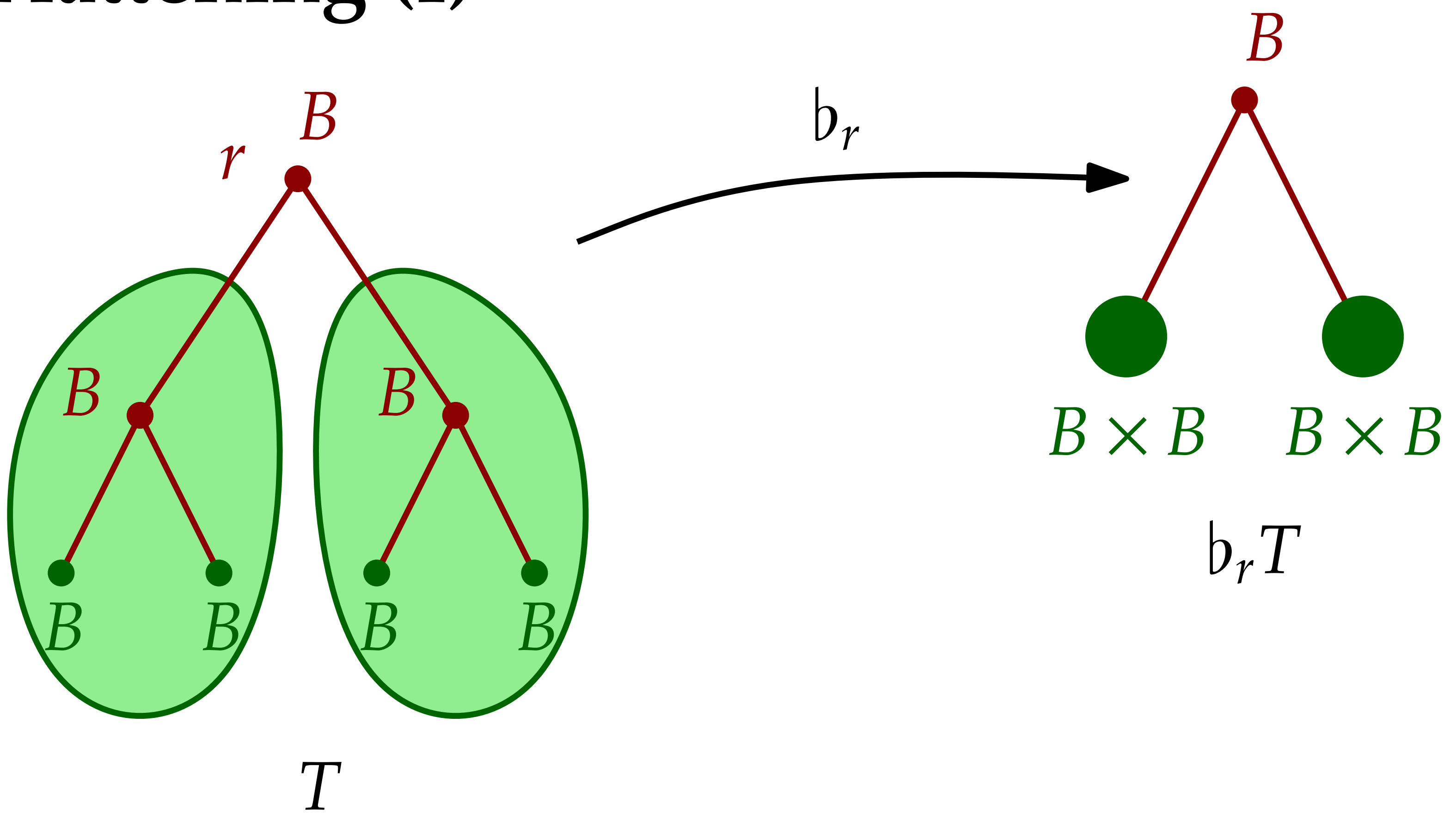

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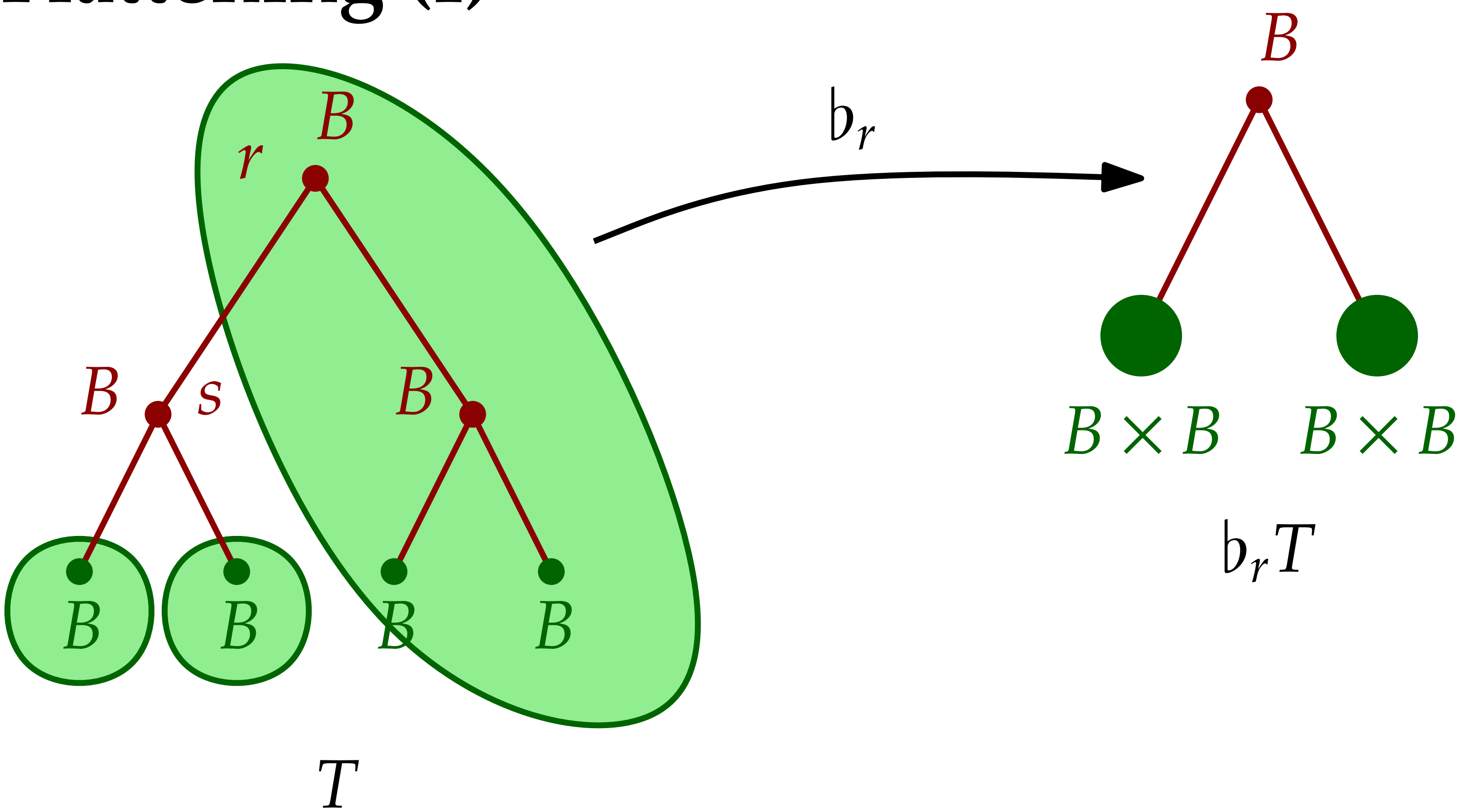

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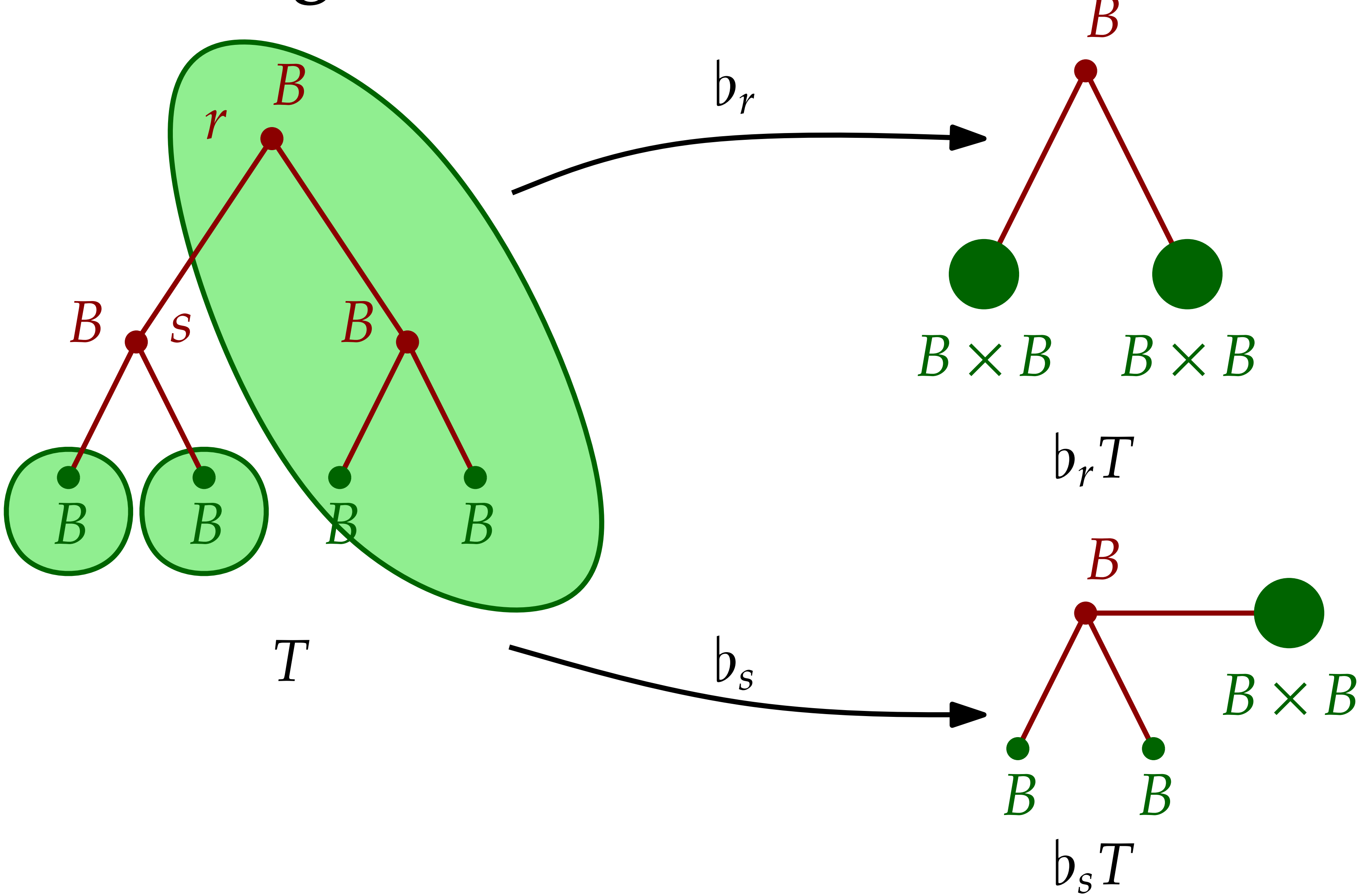

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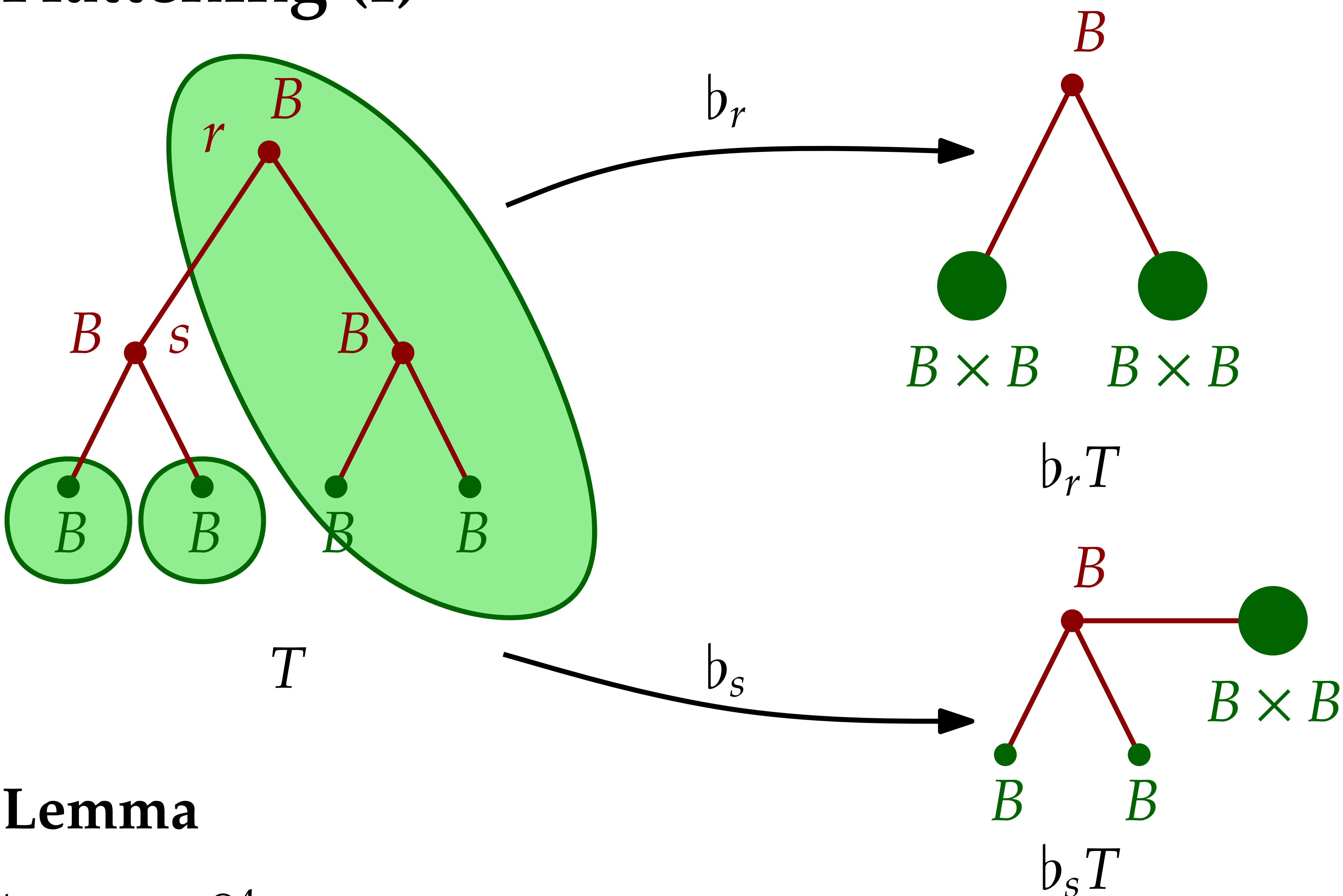

Lemma

$\flat_s : (\mathbb{C}B)^{\otimes 4} \to \mathbb{C}B \otimes \mathbb{C}B \otimes \mathbb{C}(B \times B)$
maps $\mathrm{EM}(T)$ into $\mathrm{EM}(\flat_s T)$.

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$\mathrm{EM}(T) = \bigcap_s \mathrm{EM}(\flat_s T)$

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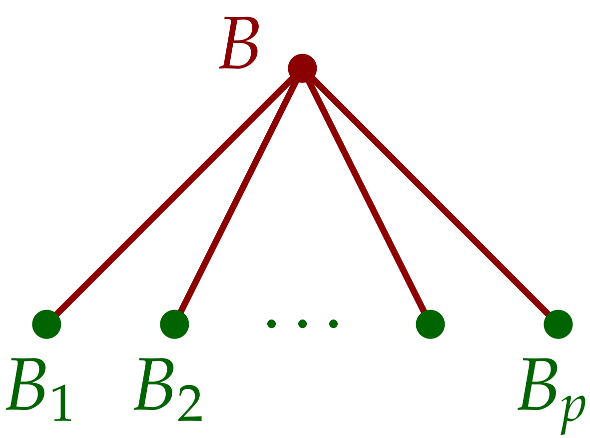

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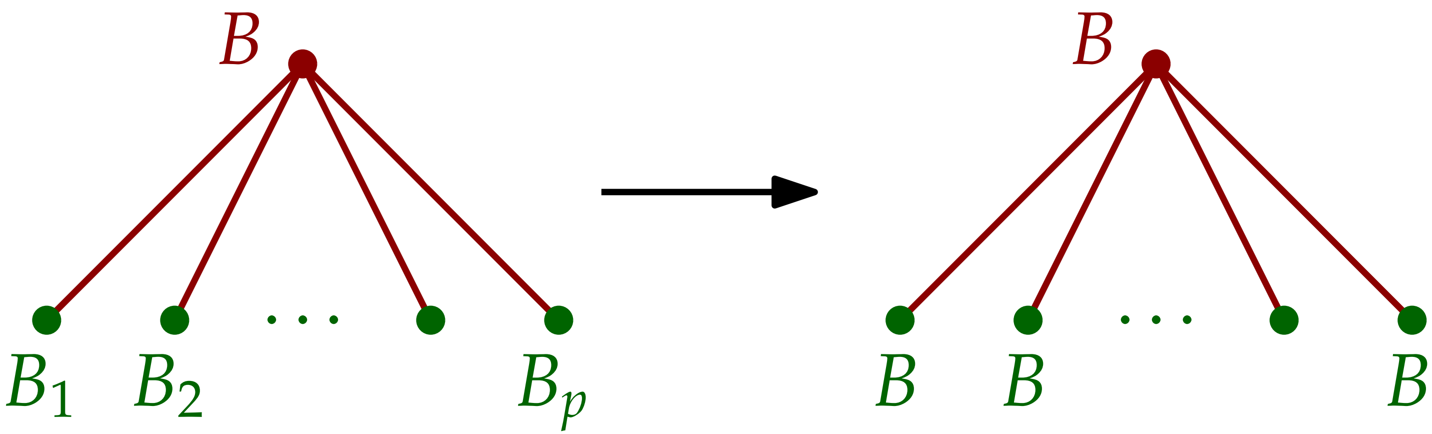
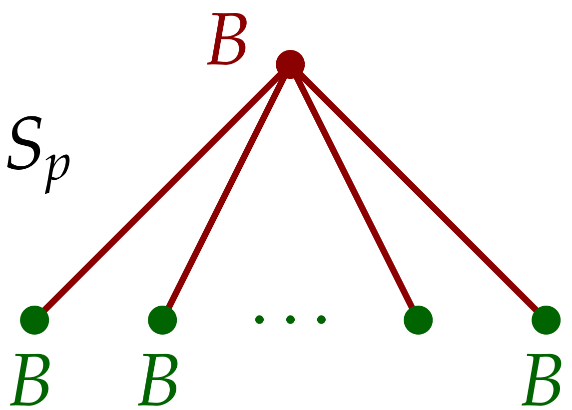


Proposition (A-R, Landsberg-Manivel, D-K)

Further reduction to B-leaved trees.

Summary so far

$V := \mathbb{C}B$ space of distributions on B
$V^{\otimes p}$ space of distributions on B^p
$\mathrm{EM}(S_p) \subseteq V^{\otimes p}$ equivariant model of S_p



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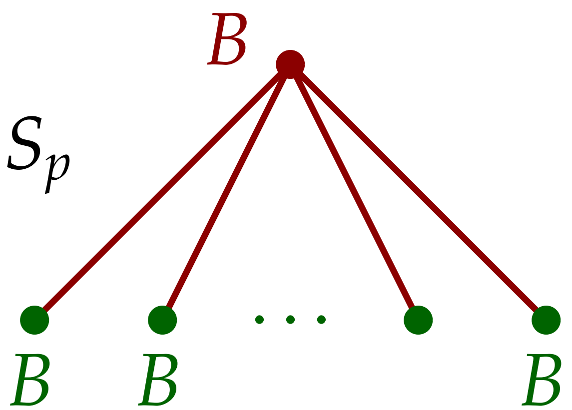

Degree bound + membership test for $\mathrm{EM}(S_p)$
$\Rightarrow$ same for $\mathrm{EM}(T)$.

$G = \{1\} \rightsquigarrow \mathrm{EM}(S_p) = \{\text{tensors of border rank} \leq |B|\}$

Example: Jukes-Cantor binary

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P lies in $\mathrm{EM}(S_p) \subseteq V^{\otimes p}$ iff

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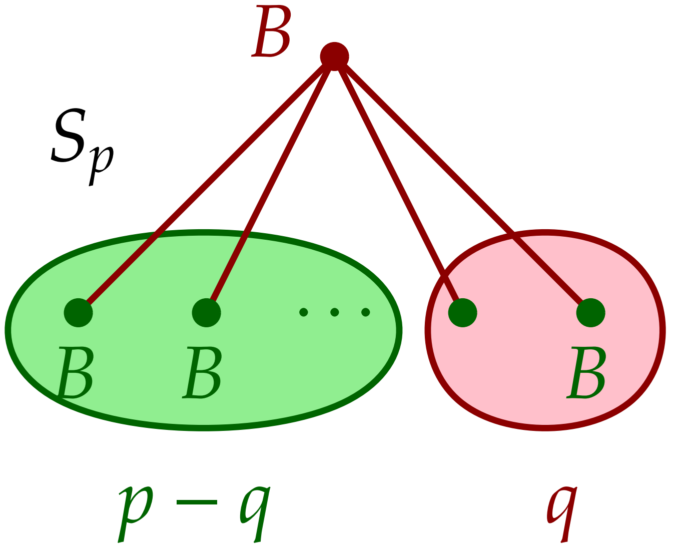
Lemma

if $P \in \mathrm{EM}(S_p)$
and Q is G-invariant
$\rightsquigarrow$ new distribution $\in \mathrm{EM}(S_{p-q})$.



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$P' := \langle P, e_0 \rangle \Rightarrow P'_w = P_{w0}$
$P' := \langle P, e_1 \otimes e_1 \rangle \Rightarrow P'_w = P_{w11}$
$\vdots$

map $\mathrm{EM}(S_p)$ into $(S_{p'})$

Contraction characterises models

$P \in V^{\otimes [p]}$
$Q \in V^{\otimes I},\ I \subseteq [p]$
$\rightsquigarrow$ *contraction* $\langle P, Q \rangle \in V^{\otimes [p] - I}$:
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implies main theorems!

Flatten and contract!

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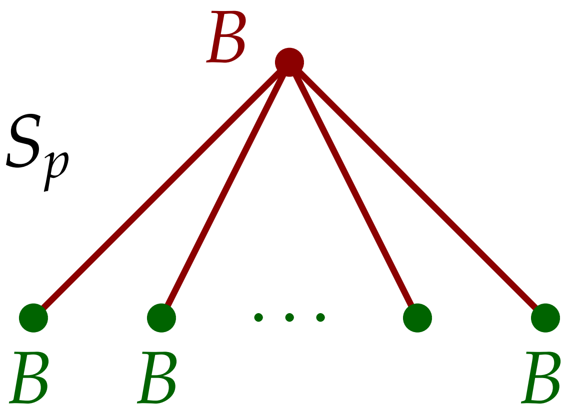

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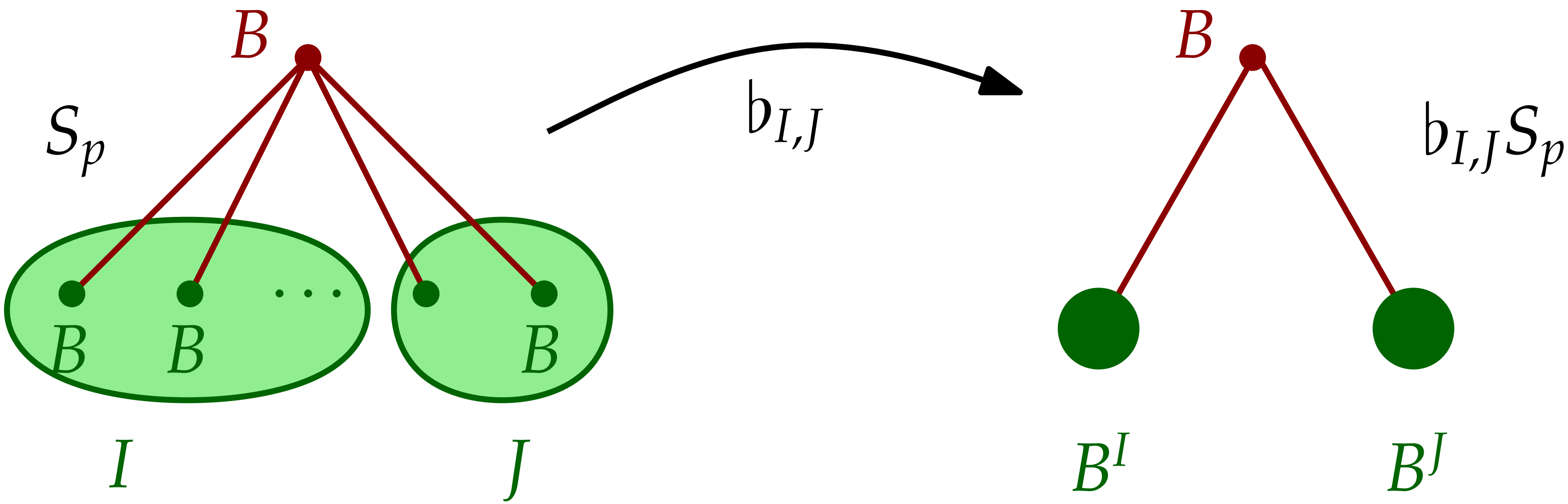

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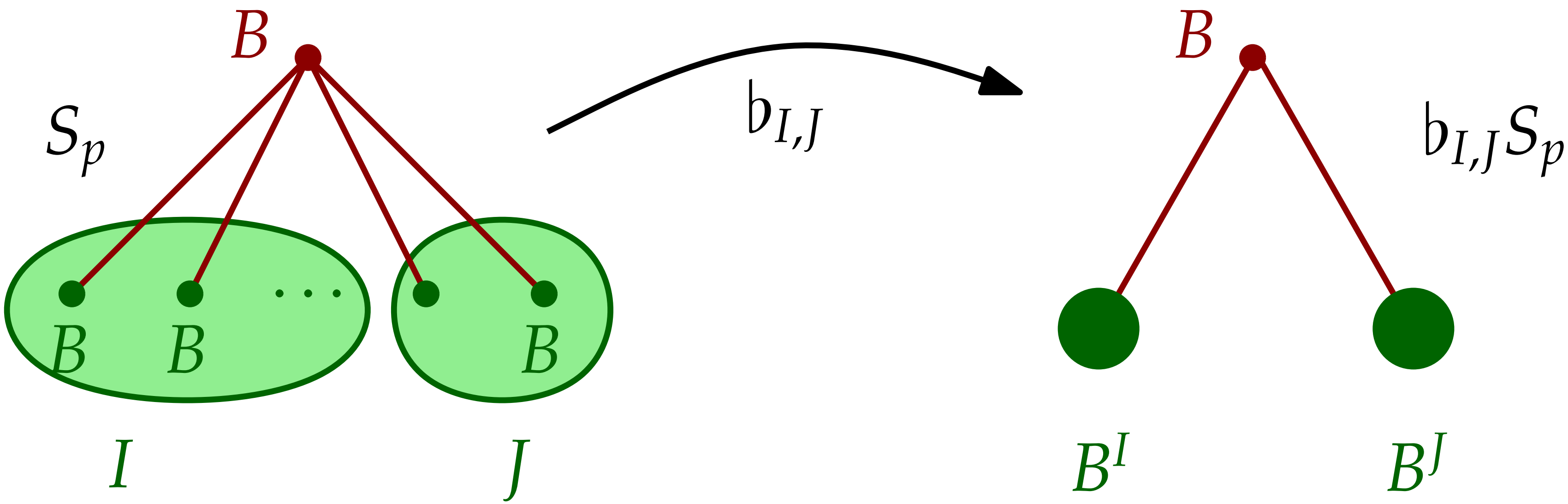

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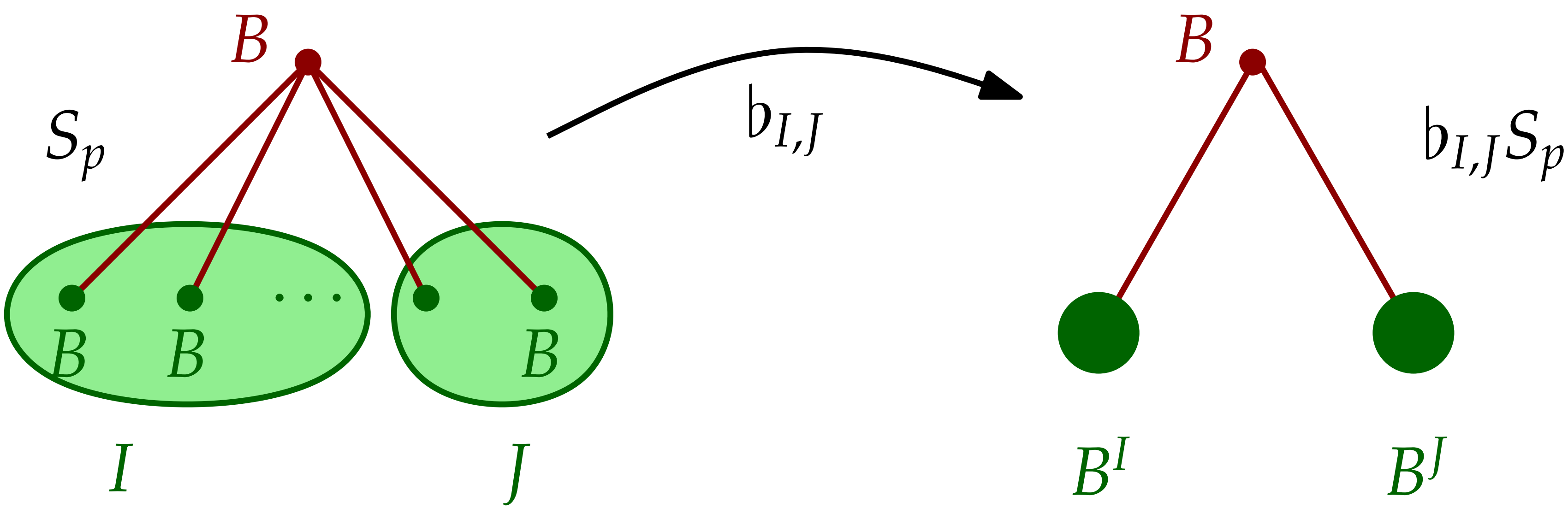

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Lemma

$\mathrm{EM}(\flat_{I,J} S_p) = \{$ G-equivariant linear maps $V^{\otimes I} \to V^{\otimes J}$ of rank $\leq k_\chi$ in component χ $\}$

$\rightsquigarrow$ determinantal equations for $\mathrm{EM}(S_p)$

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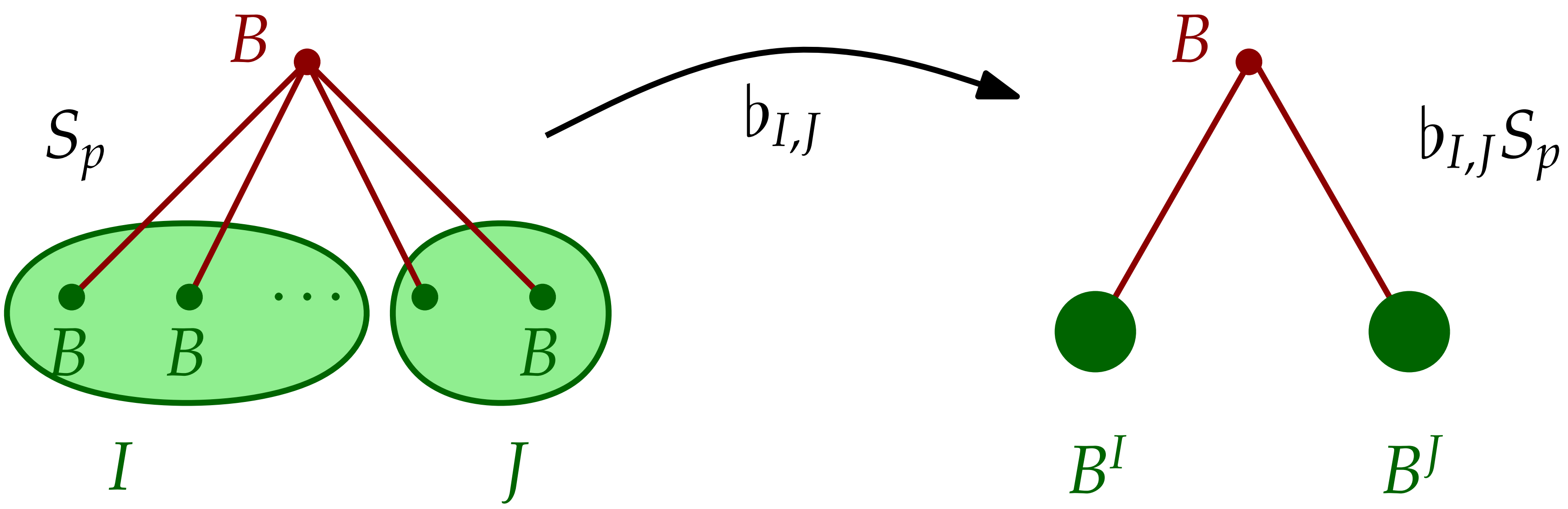

Flattening variety $Y_p := \bigcap_{I,J} \mathrm{EM}(\flat_{I,J} S_p)$

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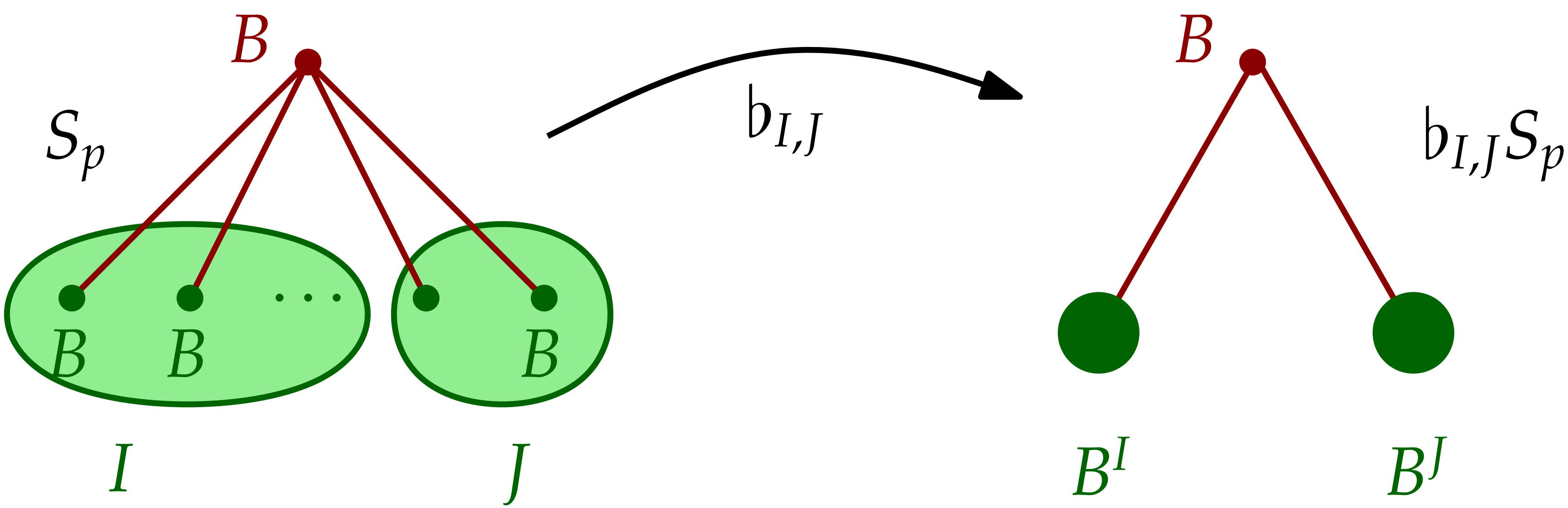

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$Y_p = \mathrm{EM}(S_p)$ for some models:

- JC binary (Sturmfels-Sullivant)
- GM binary (Landsberg-Manivel, Raicu)

Infinite tree models

$Q \in V^{\otimes q}$ G-invariant

$$V^{\otimes 0} \xleftarrow{\langle \cdot\,, Q \rangle} V^{\otimes q} \longleftarrow V^{\otimes 2q} \longleftarrow \cdots \longleftarrow V^{\otimes (\infty q)}$$

Infinite tree models

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$$
\begin{array}{ccccccccc}
 & \langle\,\cdot\,, Q\rangle & & & & & & & \\
V^{\otimes 0} & \longleftarrow & V^{\otimes q} & \longleftarrow & V^{\otimes 2q} & \longleftarrow & \cdots & \longleftarrow & V^{\otimes(\infty q)} \\
\rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & & & & \rotatebox[origin=c]{-90}{$\supseteq$} \\
Y_0 & \longleftarrow & Y_q & \longleftarrow & Y_{2q} & \longleftarrow & \cdots & \longleftarrow & Y_{\infty q}
\end{array}
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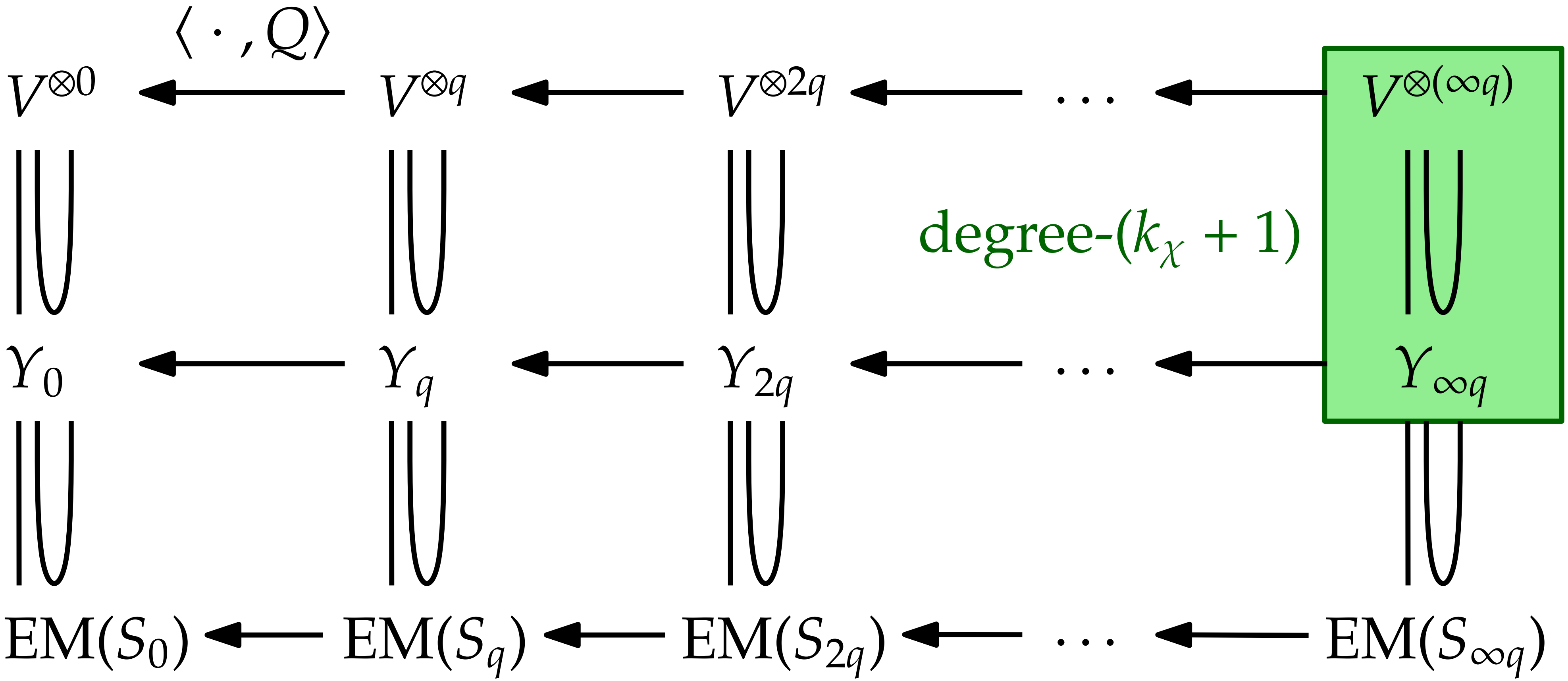
\mathrm{EM}(S_0) & \longleftarrow & \mathrm{EM}(S_q) & \longleftarrow & \mathrm{EM}(S_{2q}) & \longleftarrow & \cdots & \longleftarrow & \mathrm{EM}(S_{\infty q})
\end{array}
$$

Infinite tree models

$Q \in V^{\otimes q}$ G-invariant



Infinite tree models

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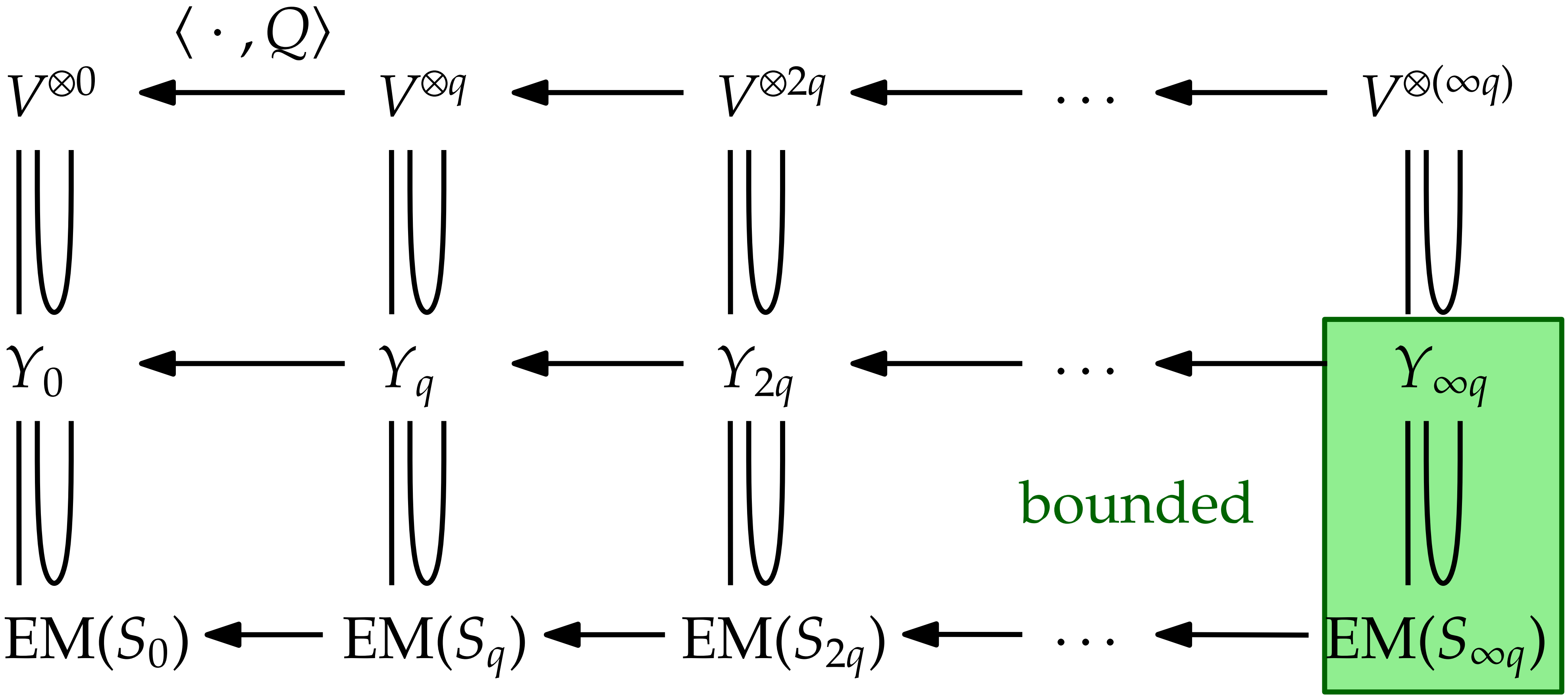

Infinite tree models, symmetries

$H_p := \mathrm{Sym}(p) \ltimes \mathrm{GL}_G(V)^p$
acts on $V^{\otimes p} \supseteq Y_p \supseteq \mathrm{EM}(S_p)$

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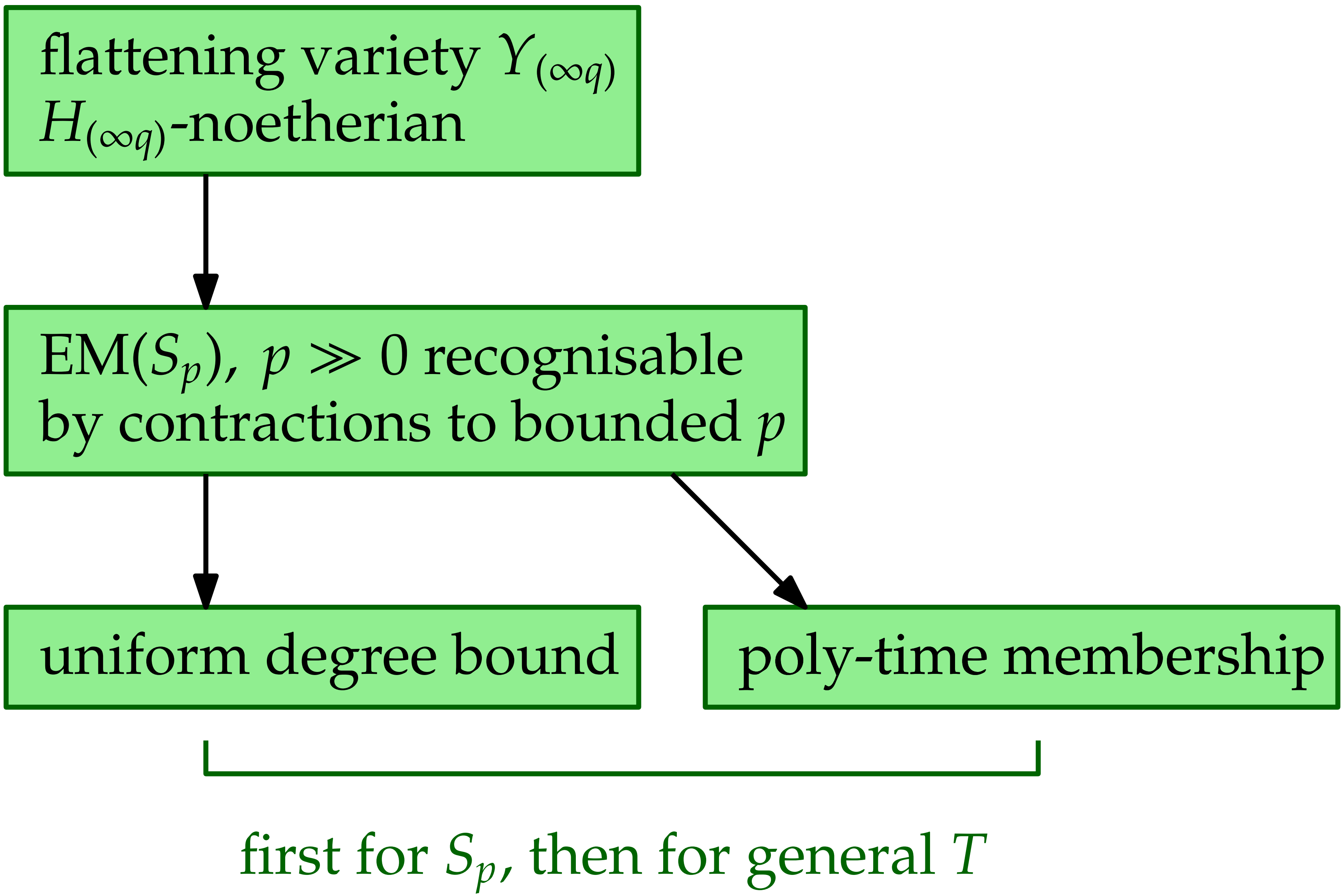
Theorem

For suitable q and Q, every $H_{(\infty q)}$-stable closed subvariety of $Y_{(\infty q)}$ is defined by finitely many $H_{(\infty q)}$-orbits of equations.

In particular for $\mathrm{EM}(S_{(\infty q)})$!

Summary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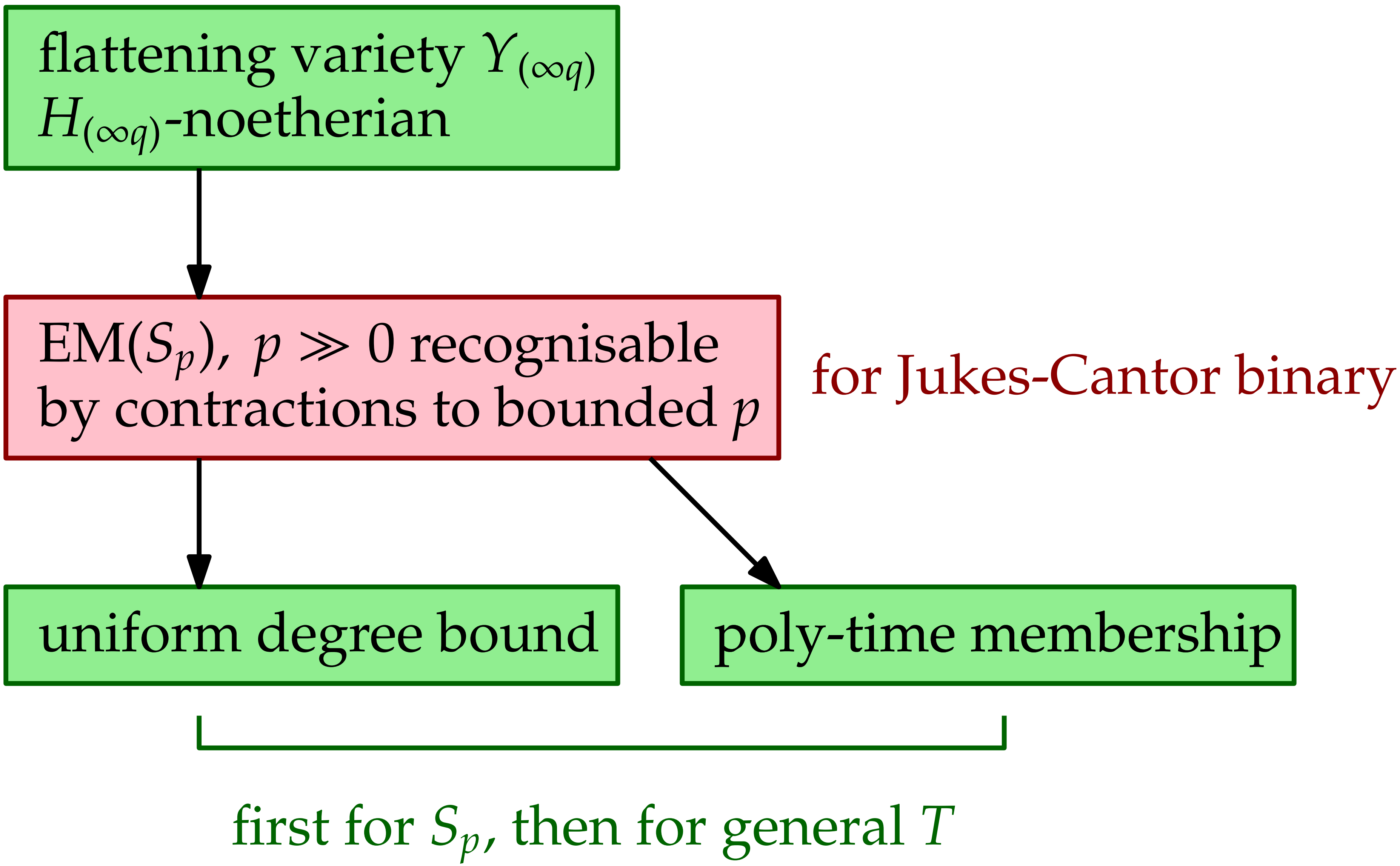
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Contraction for Jukes-Cantor binary

$G = B = \{-1, +1\}$

$V = \mathbb{C}B = \langle (-1), (1) \rangle = \langle e_0 := (-1) + (1), e_1 := (-1) - (1) \rangle$

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Theorem (Sturmfels-Sullivant)

P lies in $\mathrm{EM}(S_p) \subseteq V^{\otimes p}$ iff

- $P_w = 0$ if $w \in \{0, 1\}^p$ has odd weight
- $P_{w00} P_{u11} - P_{w11} P_{w00} = 0, P_{w01} P_{u10} - P_{w10} P_{u01} = 0$

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Equivalently: $P : V^{\otimes I} \to V^{\otimes [p]-I}$ is G-equivariant and has rank ≤ 1 in each character.

$\rightsquigarrow \mathrm{EM}(S_p) = Y_p$.

Contraction for J-C binary, continued

$V = \langle e_0, e_1 \rangle$
$P \in V^{\otimes p}$ with $p \geq 6$

Claim: $P \notin \mathrm{EM}(S_p) \Rightarrow$ some $\langle P, Q \rangle \notin \mathrm{EM}(S_{p-q})$.

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- $|J| \geq 4 \Rightarrow$ image in $V^{\otimes J}$ of $\flat_{I,J} P$ can be contracted with some e_0 or $c e_0 \otimes e_0 + d e_1 \otimes e_1$ without dropping dimension

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- $|I| = |J| = 3 \Rightarrow P$ can be contracted
  in one factor in each of I, J

Randomised algorithm for JC binary

Input

$P \in V^{\otimes p}$

Output

$P \in \mathrm{EM}(S_p)$?

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Algorithm

b :=true;
for each $I \subseteq [p]$, $|I| \geq p - 5$ do
    generate G-invariant $Q \in V^{\otimes I}$ at random;
    $b := b$ and $\langle P, Q \rangle \in \mathrm{EM}(S_{[p]-I})$;
od;
return b;

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For suitable q and Q, every $H_{(\infty q)}$-stable closed subvariety of $Y_{(\infty q)}$ is defined by finitely many $H_{(\infty q)}$-orbits of equations.

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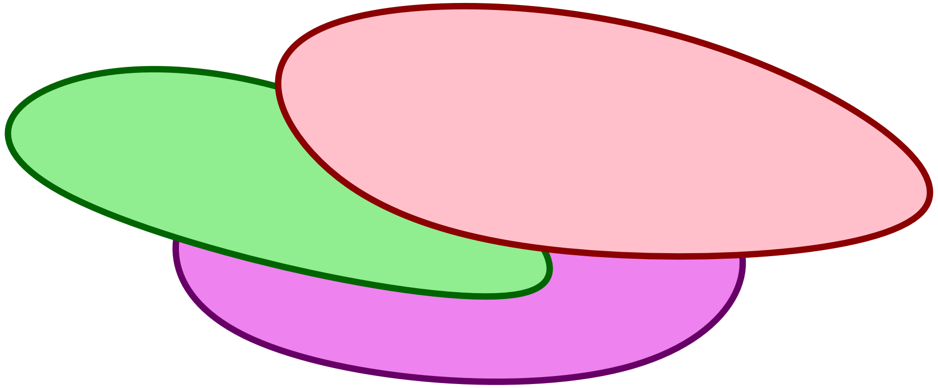
- $Y_{(\infty q)}$ itself is defined by finitely many orbits (Borel's fixed point theorem, uses G abelian)
- $\mathbb{C}^{\ell \times \mathbb{N}}$ is Sym($\mathbb{N}$) Noetherian (Cohen, Hillar-Sullivant)
- $Y_{(\infty q)}$ is covered by finitely many spaces $C^{\ell \times \mathbb{N}}$ in a Sym($\mathbb{N}$)-equivariant way.



Abelian equivariant tree models are characterised by flattening and contracting to bounded star models.