Orthogonally decomposable tensors as semisimple algebras

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Singular value decomposition

Every $A \in \mathbb{C}^{m\times n}$ can be written as $A = \sum_{i=1}^{k} u_i v_i^T$ with nonzero and pairwise perpendicular $u_1, \ldots, u_k \in \mathbb{C}^m$ and similar $v_1, \ldots, v_k \in \mathbb{C}^n$.

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If $m = n$ and A is skew, then $k = 2\ell$ and one can take $v_i = u_{i+\ell}$ for $i \leq \ell$ and $v_i = -u_{i-\ell}$ for $i > \ell$; then $A = \sum_{i=1}^{l} (u_i v_i^T - v_i u_i^T)$.

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Question. Which *tensors* admit orthogonal decompositions?

Orthogonally and unitarily decomposable tensors

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Definition. A tensor $T \in V_1 \otimes \cdots \otimes V_d$ is *odeco/udeco* if it can be written as $T = \sum_{i=1}^k v_{i1} \otimes \cdots \otimes v_{id}$ where for each $j = 1, \dots, d$ the vectors $v_{1j}, \dots, v_{kj}$ are nonzero and pairwise perpendicular.

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Definition. A symmetric tensor $T \in \mathrm{Sym}^d(V) \subseteq V^{\otimes d}$ is *symmetrically odeco/udeco* if it can be written as $T = \sum_{i=1}^{k} \pm v_i^{\otimes d}$ for nonzero, pairwise perpendicular v_i.

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Example. With $V = \mathbb{R}^2$ and $d = 3$ the tensor
$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$
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Definition. An alternating tensor $T \in \mathrm{Alt}^d V \subseteq V^{\otimes d}$ is *alternatingly odeco/udeco* if $T = \sum_{i=1}^{k} v_{i1} \wedge \cdots \wedge v_{id}$ for $k \cdot d$ nonzero, pairwise perpendicular vectors $v_{11}, \ldots, v_{kd}$.

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Main theorem. For $d \geq 3$ odeco/udeco tensors form a real-algebraic variety defined by polynomials of the following degrees:

	odeco ($\mathbb{R}$)	*udeco* ($\mathbb{C}$)
symmetric	2 (associativity)	3 (semi-associativity)
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$$\mathrm{O}_n \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}(\mathrm{Sym}^d V)$$

The lhs is compact, so the image is closed, and its pre-image in $\mathrm{Sym}^d(V) \setminus \{0\}$ is the set of nonzero sym odeco tensors. $\square$

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Proposition. For $d \geq 3$ the orthogonal decomposition is unique.

Proof (ordinary case). Contracting $T = \sum_{i=1}^k v_{i1} \otimes \cdots \otimes v_{id}$ with a general tensor in $V_3 \otimes \cdots \otimes V_d$ yields a two-tensor A with distinct nonzero singular values. □

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(*This yields an algorithm for orthogonal decomposition—Kolda.*)

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Conjecture (Robeva). This characterises ordinary odeco tensors.

Via the isomorphism $V^{\otimes 3} \cong V^* \otimes V^* \otimes V$, a $T \in \mathrm{Sym}^3(V) \subseteq V^{\otimes 3}$ gives rise to a bilinear map $V \times V \to V$, $(u, v) \mapsto u \cdot v = uv$. Note: $uv = vu$ since $(12)T = T$; and $(uv|w) = (uw|v)$ since $(23)T = T$.

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Proof. $\Rightarrow$: If $T = \sum_i u_i^{\otimes 3}$, then

$(xy)z = (\sum_i (u_i|x)(u_i|y)u_i)z = \sum_i (u_i|x)(u_i|y)(u_i|z)||u_i||^2 u_i = x(yz)$

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$\Leftarrow$ may assume $(V, \cdot)$ is simple. Pick x such that $M_x : y \mapsto xy$ is nonzero. Then $\ker M_x$ is an ideal, so 0. Define $y * z := M_x^{-1}(yz)$.
$\rightsquigarrow$ $(V, *)$ is simple, comm, ass, with 1 and compatible (.|.), so $\cong \mathbb{R}$.

□

Main theorem for ordinary odeco three-tensors

The proof is very similar, except now $T \in U \otimes V \otimes W$ gives rise to a commutative algebra structure on $U \oplus V \oplus W$ with $U \cdot V \subseteq W$, $U \cdot U = \{0\}$, etc., and we are interested only in *homogeneous* ideals.

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Partial associativity means that $(xy)z = x(yz)$ whenever x, y, z are homogeneous and x, z belong to the same space (U, V, W).

Main theorem for alternatingly odeco three-tensors

Again, $T \in \mathrm{Alt}^3(V)$ gives a bilinear multiplication $(x, y) \mapsto xy$. Now we have $xy = -yx$ and $(xy|z) = -(xz|y)$.

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Proposition. T is alternatingly odeco iff $(V, \cdot)$ satisfies the Jacobi identity and furthermore has the property that for each $x, y, z \in V$ the map $C := M_x M_{(yz)} + M_y M_{(zx)} + M_z M_{(xy)}$ centralises all M_u.

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$\Leftarrow$: $(V, \cdot)$ is then a compact Lie algebra. Their classification implies that the only simple one for which C is central, is $(\mathbb{R}^3, \times)$. $\square$

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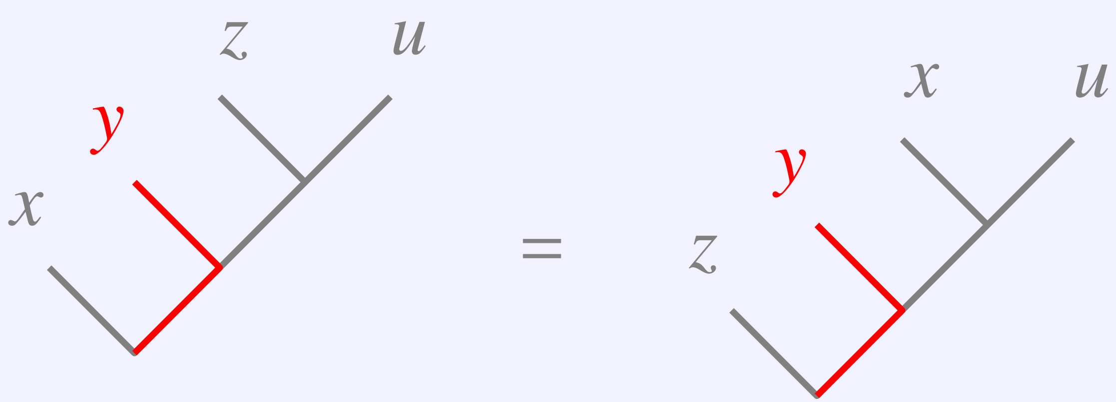

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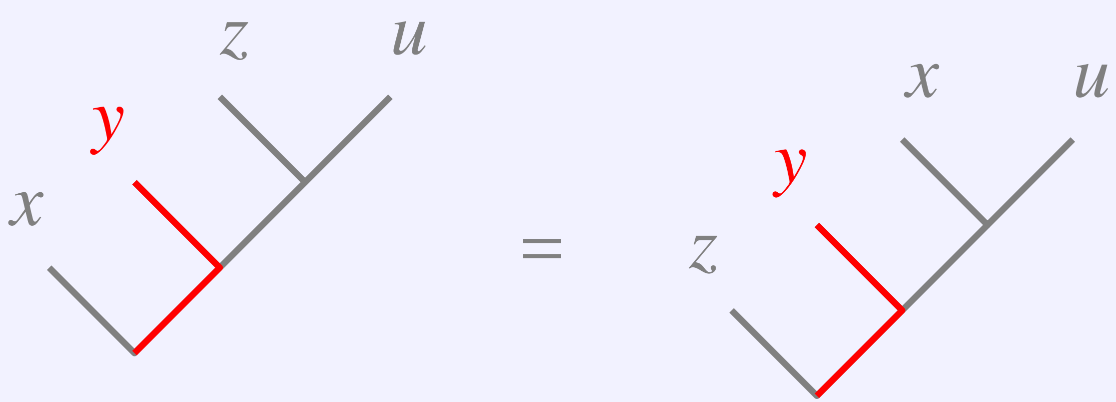

We have a similar characterisation for ordinary three-tensors.

What about tensors of order > 3?

Ordinary case. For $d \geq 4$, a tensor in $V_1 \otimes \cdots \otimes V_d$ is odeco/udeco iff its *flattening* into $(\bigotimes_{i \in I_1} V_i) \otimes \cdots \otimes (\bigotimes_{i \in I_e} V_i)$ is for each partition $I_1, \ldots, I_e$ of $\{1, \ldots, d\}$ with at least one $|I_j| > 1$.

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This proves the main theorem, *except* …

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There is a 280-dimensional space of cubic equations for udeco tensors in $\mathrm{Alt}^3\mathbb{C}^6$, one of which looks like:

$$t_{1,4,5}t_{2,3,4}\bar{t}_{1,3,5} - t_{1,3,4}t_{2,4,5}\bar{t}_{1,3,5} + t_{1,2,4}t_{3,4,5}\bar{t}_{1,3,5} + t_{1,4,6}t_{2,3,4}\bar{t}_{1,3,6} - t_{1,3,4}t_{2,4,6}\bar{t}_{1,3,6} + t_{1,2,4}t_{3,4,6}\bar{t}_{1,3,6} - t_{1,4,6}t_{2,4,5}\bar{t}_{1,5,6} + t_{1,4,5}t_{2,4,6}\bar{t}_{1,5,6} - t_{1,2,4}t_{4,5,6}\bar{t}_{1,5,6} + t_{2,4,6}t_{3,4,5}\bar{t}_{3,5,6} - t_{2,4,5}t_{3,4,6}\bar{t}_{3,5,6} + t_{2,3,4}t_{4,5,6}\bar{t}_{3,5,6}$$

…but the algebra has *no* polynomial identities of degree 3 :-(