Where innovation starts

Finiteness of the k-factor model

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A real-algebraic problem

Definition

$F_{k,n} := \{D + SS^T \mid D > 0$ diagonal, $S \in M_{n,k}(\mathbb{R})\}$
k-factor model with n observed variables

Observation

$Y \in F_{k,n} \Rightarrow Y[I] \in F_{k,|I|}$ for all $I \subseteq \{1, \ldots, n\}$.

Question

$\exists ? n_0 = n_0(k)$ such that for $n \geq n_0$:
$Y \in F_{k,n} \Leftrightarrow Y[I] \in F_{k,n_0}$ for all $I, |I| = n_0$.

Theorem (Drton-Xiao 2008)

Yes for $k = 1$ $(n_0 = 4)$ and $k = 2$ $(n_0 = 6)$.
Open in general.

The Gaussian k-factor model

Density

$X_1, \ldots, X_n, Z_1, \ldots, Z_k$ jointly Gaussian, mean 0

$$f_{X,Z}(x,z) = \frac{1}{\sqrt{(2\pi)^{(n+k)}\det(\Sigma)}} \exp\left(-\frac{1}{2}(x,z)\Sigma^{-1}(x,z)^T\right)$$

Conditions

- Σ positive definite; and
- the X_i (“observed”) pairwise independent given $Z_1, \ldots, Z_k$ (“hidden”).

Projection to $M_n(\mathbb{R})$ gives parameter space $F_{k,n}$.

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Ideal approach to model validation

Definition

$\overline{F_{k,n}} \subseteq M_n(\mathbb{C})$ Zariski closure

Observation

$Y \in \overline{F_{k,n}} \Rightarrow Y[I] \in \overline{F_{k,|I|}}$ for all $I \subseteq \{1, \ldots, n\}$.

Main Theorem

$\exists n_1 = n_1(k)$ such that for $n \geq n_1$:
$Y \in \overline{F_{k,n}} \Leftrightarrow Y[I] \in \overline{F_{k,n_1}}$ for all $I, |I| = n_1$.

Disclaimer

- No upper bound on n_1 yet, except $n_1(1) = 4$ (Loera-Sturmfels-Thomas 1995).
- Only set-theoretically so far.

Methods

Definition

G group acting on ring R
R is G-*Noetherian* if every chain $I_1 \subseteq I_2 \subseteq \ldots$ of G-stable ideals stabilises.

Theorem (Aschenbrenner-Hillar-Sullivant 2007)

$\mathbb{C}[x_{ij} \mid i = 1, \ldots, k; j \in \mathbb{N}]$ is $\mathrm{Sym}(\mathbb{N})$-Noetherian.

Non-example

$\mathbb{C}[y_{ij} \mid i, j \in \mathbb{N}]$ is not $\mathrm{Sym}(\mathbb{N})$-Noetherian.

Methods, continued

Definition

G group acting on topological space X
X is G-*Noetherian* if every chain $X_1 \supseteq X_2 \supseteq \ldots$ of G-stable closed subsets stabilises.

Important lemma

$H \subseteq G$ and X an H-Noetherian space
$\Rightarrow G \times_H X$ a G-Noetherian space.

Reformulation

Theorem

$$\tilde{F}_{k,\mathbb{N}} := \{Y \in M_{\mathbb{N},\mathbb{N}}(\mathbb{C}) \mid \det(Y[I,J]) = 0 \text{ for all disjoint } I, J \subseteq \mathbb{N}, |I| = |J| = k+1\}$$

is a $\mathrm{Sym}(\mathbb{N})$-Noetherian topological space.

Observation

This implies the Main Theorem:

- $\tilde{F}_{k,\mathbb{N}}$ is cut out by one $\mathrm{Sym}(\mathbb{N})$-orbit of equations;
- $\overline{F_{k,\mathbb{N}}}$ is a $\mathrm{Sym}(\mathbb{N})$-stable closed subset of $\tilde{F}_{k,\mathbb{N}}$, hence cut out by finitely many further orbits of equations.

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Proof sketch

Induction on k

- $\tilde{F}_{0,\mathbb{N}}$ consists of all diagonal matrices, statement follows from Aschenbrenner-Hillar.
- $\tilde{F}_{k,\mathbb{N}} \subseteq \tilde{F}_{k-1,\mathbb{N}} \cup Z$, where Z is again $\mathrm{Sym}(\mathbb{N})$-Noetherian.

Construction of Z

$Z \ni$ all elements of $\tilde{F}_{k,\mathbb{N}}$ having some invertible off-diagonal $k \times k$-minor.
First fix the position of this minor to $I = \{1, \ldots, k\}$ and $J = \{k+1, \ldots, 2k\}$; later move around with $\mathrm{Sym}(\mathbb{N})$.

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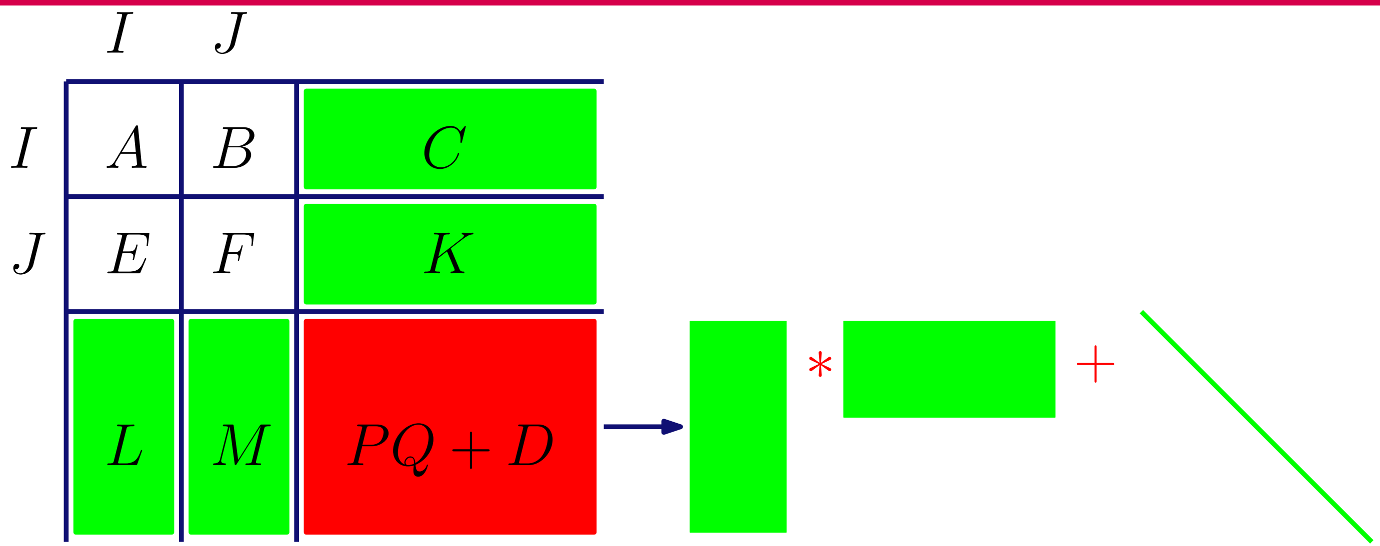
Proof sketch, continued



is an $H = \mathrm{Sym}(\mathbb{N} \setminus (I \cup J))$-equivariant expression in $(A, B, C, D, E, F, K, L, M, P, Q) \in X$;

X is $\mathrm{Sym}(\mathbb{N} \setminus (I \cup J))$-Noetherian by Aschenbrenner-Hillar-Sullivant (and Hilbert);

take Z the image of $\mathrm{Sym}(\mathbb{N}) \times_{\mathrm{Sym}(\mathbb{N} \setminus (I \cup J))} X$ $\square$

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References

Arxiv

Draisma, keyword: finiteness, 2008
Drton-Sturmfels-Sullivant, keyword: factor, 2007