Finiteness for polynomial functors

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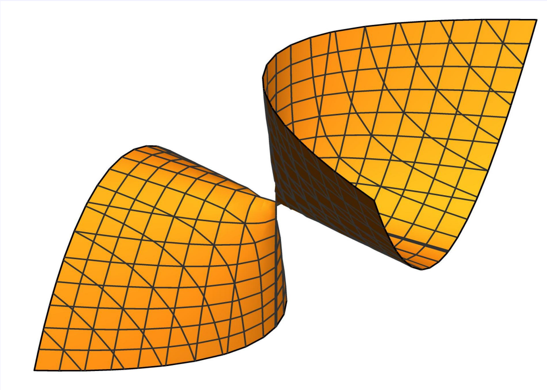

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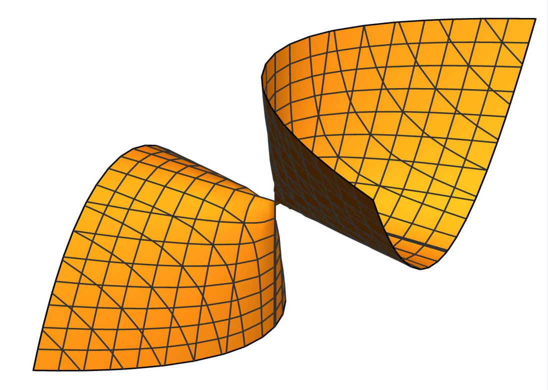

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X_n is the solution set to $\{x_{ij}x_{kl} - x_{il}x_{kj} = 0 \mid i \neq k, j \neq l\}$.

For all $n \geq 2$, the equations for X_n are found from the equation for X_2 by applying symmetries of the form $a \mapsto gag^T$. (if char $\neq 2$)

Observation

If $\pi : A_m \to A_n$ is a linear map with $\pi(X_m) \subseteq X_n$, and if f is an equation for X_n, then $f \circ \pi$ is an equation for X_m of degree $(\leq) \deg f$.

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General message
Often, the equations of finitely many varieties pull back to yield sufficient equations for defining all varieties in the sequence.

In particular, this holds for varieties in polynomial functors!

What's a polynomial functor?

Polynomial functors P:

- behave like univariate polynomials;
- take as argument a finite-dimensional vector space V;
- return as value a finite-dimensional vector space $P(V)$;
- take a linear $\varphi : V \to W$ to a linear $P(\varphi) : P(V) \to P(W)$;
- in such a manner that $P(1_V) = 1_{P(V)}$ and $P(\psi \circ \varphi) = P(\psi) \circ P(\varphi)$;
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Trivial example

$V \mapsto V$, a polynomial functor of degree 1.

Running example

$A : V \mapsto \{a \in V \otimes V \mid a^T = a\}$, of degree 2.

Setting
K an infinite field
Vec the category of finite-dimensional K-vector spaces

Definition
$P : \mathbf{Vec} \to \mathbf{Vec}$ is *polynomial* of degree $\leq d$ if $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}(P(V), P(W))$ is polynomial of degree $\leq d$ for all V, W.

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Examples
- $P(V) = V^{\otimes d}$ with $P(\varphi)v_1 \otimes \cdots \otimes v_d = (\varphi v_1) \otimes \cdots \otimes (\varphi v_d)$
- $P(V) = S^d V = V^{\otimes d} / \langle \{ v_1 \otimes \cdots \otimes v_d - v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)} \} \rangle$
- char $K = p > 0$, K perfect, $P(V) = S^p(V) / \{ f^p \mid f \in V \}$

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Homogeneous decomposition

$P = P_0 \oplus \cdots \oplus P_d$ with $P_e = \{q \in P(V) \mid \forall t \in K : \; P(t 1_V) q = t^e q\}$

Subvarieties of polynomial functors

$P \rightsquigarrow$ a functor **Vec** $\to$ **Top**, where $P(V)$ has the Zariski topology

Definition

A **Vec**-*subvariety* X of P assigns to each V a subvariety $X(V) \subseteq P(V)$, such that for all $\varphi : V \to W$ the linear map $P(\varphi) : P(V) \to P(W)$ maps $X(V)$ into $X(W)$.

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- rank-one tensors: $X(V) = \{v_1 \otimes \cdots \otimes v_d\} \subseteq V^{\otimes d}$ (Segre)
- d-th powers of linear forms: $X(V) = \{v^d\} \subseteq S^d V$ (Veronese)
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Main Theorem (Noetherianity for polynomial functors)

For any **Vec**-subvariety X in a polynomial functor P of finite degree, $\exists$ a V_0 such that $\forall V$: $X(V)$ is defined by the pull-backs of equations for $X(V_0)$ under linear maps $P(\varphi)$ for $\varphi : V \to V_0$.

Case 1: char $K \neq 2$

Actually, any **Vec**-subvariety $X \subseteq A : V \mapsto \{a \in V \otimes V \mid a^T = a\}$ consists of matrices of rank at most k for some $k \in \{0, 1, 2, \ldots, \infty\}$.

So either $X = A$ or we can take $V_0 = K^{k+1}$, which is the smallest dimension for which you see a $(k+1) \times (k+1)$-determinant.

Proof in running example

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Case 2: char $K = 2$

Then A is not an irreducible polynomial functor, as it contains $B : V \mapsto \{a \in V \otimes V \mid a^T = a, \forall x \in V^* : x^T a x = 0\}$, the nonzero subspace of (skew-)symmetric matrices with zeroes on the diagonal. Now first prove the theorem for A/B, then lift to A.

Consequences and special cases 1

Known before

• degree ≤ 2 (tuples of matrices, Eggermont 2014)
• for S^3V (cubics, Derksen-Eggermont-Snowden 2016)
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Slice rank

$X_1(V) := \{T \in V^{\otimes d} \mid \exists i \in [d], v \in V, S \in \bigotimes_{j \neq i} V : T = v \otimes S\}$

$X_k := X_1 + \cdots + X_1$ tensors of *slice rank* $\leq k$

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The Main Theorem implies: $X_k(V)$ is defined by equations of bounded degree independent of V.

For $k = 2$ and $d = 3$ this degree is 6 *(Oosterhof).*

Twisted commutative algebras

$P \rightsquigarrow$ contravariant functor $V \mapsto K[P(V)]$ from **Vec** to K-algebras

Over $K = \mathbb{C}$, this is a *twisted commutative algebra* (Sam-Snowden). The Main Theorem implies that finitely generated tcas are topologically Noetherian.

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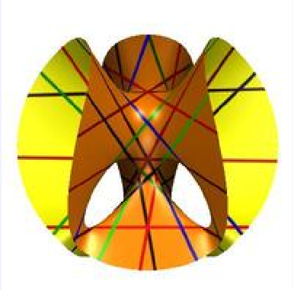
Variants of Stillman's conjecture [Erman-Sam-Snowden]

Let c be any natural number, and fix degrees $d_1, \ldots, d_k$. Then the number of codimension-c linear subspaces of $\mathbb{P}^n$ contained in a projective variety defined by k polynomials of degrees $d_1, \ldots, d_k$ is either infinite or at most some *which doesn't depend on n!*

(*Uses the main theorem for* $\bigoplus_{i=1}^{k} S^{d_i}(V)$.)



For $k = 1, d_1 = 3, c = 2$, N is at least 27:

The shift functor

U a fixed vector space $\rightsquigarrow S_U : \mathbf{Vec} \to \mathbf{Vec}, V \mapsto U \oplus V$

Observation

If P is a polynomial functor of degree d, then $P \circ S_U$ is also a polynomial functor of degree d, and $P_d \cong (P \circ S_U)_d$.

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Example

$S^d(U \oplus V) = \bigoplus_{e=0}^{d} S^{d-e}U \otimes S^e V = S^d V + \cdots$

But note that $(P \circ S_U)_e$ *typically is larger than* P_e*!*

Proof ingredients

A "lexicographic order"
Define $Q \prec P$ if $Q \not\cong P$ and for the largest e with $Q_e \not\cong P_e$ the former is a homomorphic image of the latter.

This is a well-founded order on finite-degree polynomial functors. We do induction, and assume the theorem holds for all $Q \prec P$.

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Splitting of a term of highest degree
Let $R \subseteq P_d$ be an irreducible subfunctor, and $\pi : P \to Q := P/R$.

$Q \prec P$ but we'll need other functors smaller than P

For $X \subseteq P$ let X_Q be the closure of the image in Q. Think of X as a variety over X_Q. Accordingly, $\mathcal{I}_X(V)$ is the ideal of X in $K[\pi(V)^{-1}(X_Q(V))] \cong K[X_Q(V)] \otimes K[R(V)]$ (non-canonically).

Another well-founded order
Define $\delta_X \in \{1, 2, \ldots, \infty\}$ as the minimal degree of a nonzero polynomial in $\mathcal{I}_X(V)$ over all V.

For $X, Y \subseteq P$ say $X > Y$ if $X_Q \supsetneq Y_Q$ or $X_Q = Y_Q$ and $\delta_X > \delta_Y$. As $Q \prec P$, Q is Noetherian by the induction hypothesis, so this is a well-founded order.

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Second induction hypothesis
All $Y < X$ are Noetherian.

Now if $\delta_X = \infty$, X Noetherian. So assume $\delta_X \in \mathbb{Z}_{\geq 1}$.

Proof steps

- Take $f \in \mathcal{I}_X(U)$ nonzero, homogeneous of degree δ_X.
- Pick an $r_0 \in R(U)$ such that $h := \partial_{r_0} f$ is nonzero.

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- Define $Z(V) := X(V) \setminus Y(V)$. Goal: also Z is Noetherian.
- Define $X' := X \circ S_U, P' := P \circ S_U$; then $Q' := P'/R \prec P$, so Q' is Noetherian.

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- Define $Z'(V) := \{q \in X'(V) \mid h(P(\pi_U)q) \neq 0\} \subseteq Z(U \oplus V)$.
- Prove that the projection $P' \to Q'$ restricts to a closed embedding $Z' \to \{h \neq 0\}$. Then Q' Noeth $\Rightarrow Z'$ Noeth $\Rightarrow Z$ Noetherian. $\square$