

Maximum likelihood duality for determinantal varieties

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High school probability

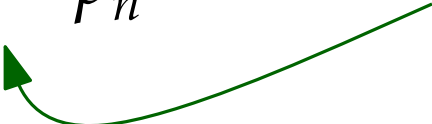
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 \rightsquigarrow prob of $U = (u_1, \dots, u_n) \in \mathbb{N}^n$ with $u_1 + \dots + u_n = N$ is

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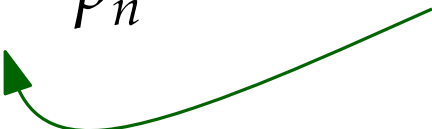
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Basic statistical problem

Given U , maximise $\ell_U(P)$ subject to constraints on P .

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Example

If only constraints are $\sum_i p_i =: p_+ = 1$ and $p_i \geq 0$
 \rightsquigarrow maximum attained by $p_i := u_i/N$.

(Mixtures of) independence

$P = (p_{ij})_{ij} \in \mathbb{R}^{m \times n}$ joint probabilities of *two* random variables
 \rightsquigarrow *independent* if $p_{ij} = q_i t_j \Leftrightarrow \text{rk}(P) = 1$

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ML-problem for independence model

Maximise $\ell_U(P)$ subject to $p_{ij} \geq 0, p_{++} = 1, \text{rk}(P) = 1$.

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$$p_{ij} = u_{i+} u_{+j} / (u_{++}^2)$$

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Mixture of r copies of independence

P convex combination of P_1, \dots, P_r as above

$\rightsquigarrow p_{++} = 1$ and $\text{rk}(P) \leq r$

ML-problem much harder!

Critical points

ML-problem for manifold $M \subseteq (\mathbb{R}_{>0})^n$

Maximise $\ell_U(P) = \prod_i p_i^{u_i}$ subject to $P \in M$.

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Derivative

$$(d_P \ell_U)(X) = \ell_U(P) \sum_i \frac{x_i}{p_i} u_i, X \in T_P M$$

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Necessary condition

P critical: $d_P \ell_U$ vanishes identically on $T_P M$

$$\Leftrightarrow \sum_i \frac{x_i}{p_i} u_i = 0 \text{ for all } X \in T_P M \Leftrightarrow (p_1^{-1}, \dots, p_n^{-1}) T_P M \subseteq U^\perp$$

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Measure of complexity(?)

Count number of critical points. . . easier over \mathbb{C} !

ML-degree

Setting

$M \subseteq (\mathbb{C}^*)^n$ smooth subvariety (locally closed) $\rightsquigarrow \text{Crit}(M) := \{(P, U) \in M \times \mathbb{P}^{n-1} \mid P^{-1}T_P M \subseteq U^\perp\}$ *variety of critical points*

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Theorem (Huh, 2012)

M closed in addition to smooth (*very affine*)

\rightsquigarrow ML-degree is signed Euler characteristic of M .

ML-degrees for matrices of given rank

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Theorem (Hauenstein-Rodriguez-Sturmfels, 2012)

For small $r \leq m \leq n$ ML-degree of M is as follows:

		(m, n)						
		$(3, 3)$	$(3, 4)$	$(3, 5)$	$(4, 4)$	$(4, 5)$	$(4, 6)$	$(5, 5)$
r	1	1	1	1	1	1	1	1
	2	10	26	58	191	843	3119	6776
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Conjecture (HRS)

$$\text{ML-degree}(M_r) = \text{ML-degree}(M_{m-r+1})$$

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Remark

- $\ell_U(P)\ell_U(Q')$ independent of P
- P positive real \Leftrightarrow so is Q'
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“ M_r and M_{m-r+1} are ML-dual”

Margins

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P satisfies $P \ker P \subseteq \text{im} P$

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$$\sum_{ij} q_{ij} = \sum_{ij} y_{ij} \frac{u_{ij}}{p_{ij}} = \sum_{ij} c p_{ij} \frac{u_{ij}}{p_{ij}} = (u_{++})^3$$



Rank of Q

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$T_P M_r$ is spanned by rank-one matrices vw^T with $(v \in \text{im} P \text{ or } w \perp \ker P)$ and $(v \perp \mathbf{1} \text{ or } w \perp \mathbf{1})$.

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For all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$ with $(x \in V \text{ or } y \perp W)$ and $(x \perp \mathbf{1} \text{ or } y \perp \mathbf{1})$ we have $x^T R^{-1} P K^{-1} y = 0$.

Criticality of Q

$P\mathbf{1}, U\mathbf{1} = R\mathbf{1}$ proportional (and so are $\mathbf{1}^T P, \mathbf{1}^T U = \mathbf{1}^T K$)

$$Q = R \cdot \left(\frac{U}{P}\right) \cdot K, \text{rk} Q = s \leq m - r + 1,$$

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E.g., $x \in V$ and $x \perp \mathbf{1} \rightsquigarrow P K^{-1} y = c R \mathbf{1} + v$ with $v \in \text{im} P \cap \mathbf{1}^\perp$
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In particular for $x \in \text{im} Q$ or $y \perp \ker Q \rightsquigarrow Q'$ critical in M_s !

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Theorem (D-Rodriguez, 2012)

M_r and M_{m-r+1} are ML-dual.



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Further work

symmetric/alternating matrices!

tensors? other ML-dual pairs of varieties?