Maximum likelihood duality for determinantal varieties

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High school probability

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$\rightsquigarrow$ prob of $U = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $u_1 + \ldots + u_n = N$ is

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Given U, *maximise* $\ell_U(P)$ subject to constraints on P.

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Example
If only constraints are $\sum_i p_i =: p_+ = 1$ and $p_i \geq 0$
$\rightsquigarrow$ maximum attained by $p_i := u_i/N$.

(Mixtures of) independence

$P = (p_{ij})_{ij} \in \mathbb{R}^{m \times n}$ joint probabilities of *two* random variables
$\rightsquigarrow$ *independent* if $p_{ij} = q_i t_j \Leftrightarrow \mathrm{rk}(P) = 1$

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Maximise $\ell_U(P)$ subject to $p_{ij} \geq 0, p_{++} = 1, \mathrm{rk}(P) = 1$.

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Mixture of r copies of independence
P convex combination of $P_1, \ldots, P_r$ as above
$\rightsquigarrow p_{++} = 1$ and $\operatorname{rk}(P) \leq r$ *ML-problem much harder!*

Critical points

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Maximise $\ell_U(P) = \prod_i p_i^{u_i}$ subject to $P \in M$.

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P critical: $d_P \ell_U$ vanishes identically on $T_P M$
$\Leftrightarrow \sum_i \frac{x_i}{p_i} u_i = 0$ for all $X \in T_P M \Leftrightarrow (p_1^{-1}, \ldots, p_n^{-1}) T_P M \subseteq U^\perp$

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Measure of complexity(?)
Count number of critical points. . . easier over $\mathbb{C}$!

ML-degree

Setting

$M \subseteq (\mathbb{C}^*)^n$ smooth subvariety (locally closed) $\rightsquigarrow$ Crit(M):= $\{(P, U) \in M \times \mathbb{P}^{n-1} \mid P^{-1}T_PM \subseteq U^\perp\}$ *variety of critical points*

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Theorem (Huh, 2012)
M closed in addition to smooth (*very affine*)
$\rightsquigarrow$ ML-degree is signed Euler characteristic of M.

ML-degrees for matrices of given rank

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For small $r \leq m \leq n$ ML-degree of M is as follows:

		(m,n)						
		$(3,3)$	$(3,4)$	$(3,5)$	$(4,4)$	$(4,5)$	$(4,6)$	$(5,5)$
	1	1	1	1	1	1	1	1
	2	10	26	58	191	843	3119	6776
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Conjecture (HRS)
ML-degree (M_r) = ML-degree (M_{m-r+1})

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Remark

- $\ell_U(P)\ell_U(Q')$ independent of P
- P positive real $\Leftrightarrow$ so is Q'
- $\ell_U(Q')$ decreases with increasing $\ell_U(P)$

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"M_r and M_{m-r+1} are ML-dual"

Margins

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$T_P M_r = \{X \in \mathbb{C}^{m\times n} \mid x_{++} = 0, X \ker P \subseteq \mathrm{im} P\}$

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$\sum_{ij} q_{ij} = \sum_{ij} y_{ij} \frac{u_{ij}}{p_{ij}} = \sum_{ij} c p_{ij} \frac{u_{ij}}{p_{ij}} = (u_{++})^3$

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$\rightsquigarrow \mathrm{rk}Q =: s \leq m - r + 1$

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Proposition
For all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$ with ($x \in V$ or $y \perp W$) *and* ($x \perp \mathbf{1}$ or $y \perp \mathbf{1}$) we have $x^T R^{-1} P K^{-1} y = 0$.

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For all $x \in \mathbb{C}^m, y \in \mathbb{C}^n$ with ($x \in V$ or $y \perp W$) *and* ($x \perp \mathbf{1}$ or $y \perp \mathbf{1}$) we have $x^T R^{-1} P K^{-1} y = 0$.

E.g., $x \in V$ and $x \perp \mathbf{1} \rightsquigarrow PK^{-1}y = cR\mathbf{1} + v$ with $v \in \mathrm{im} P \cap \mathbf{1}^\perp$
$\rightsquigarrow x^T R^{-1} P K^{-1} y = c x^T \mathbf{1} + x^T R^{-1} v = 0 + 0 = 0$ $\square$

Criticality of Q

$P\mathbf{1}, U\mathbf{1} = R\mathbf{1}$ proportional (and so are $\mathbf{1}^T P, \mathbf{1}^T U = \mathbf{1}^T K$)
$Q = R \cdot (\frac{U}{P}) \cdot K$, $\mathrm{rk}Q = s \leq m - r + 1$,
$\ker Q \supseteq K^{-1}(\ker P + \mathbb{C}\mathbf{1})^{\perp} =: W$
$\mathrm{im}Q \subseteq (R^{-1}(\mathrm{im}P \cap \mathbf{1}^{\perp}))^{\perp} =: V$

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In particular for $x \in \mathrm{im}Q$ or $y \perp \ker Q \rightsquigarrow Q'$ critical in M_s!

Bijection

$$\mathrm{Crit}(M_r) \overset{\psi_r}{\dashrightarrow} \mathrm{Crit}(M_{f(r)}),\ f(r) = s \leq m - r + 1$$

$$(P, U) \dashrightarrow ((R \cdot \tfrac{U}{P} \cdot K)/(u^3_{++}), U)$$

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Observation

- ψ_r injective
- ψ_r dominant (both spaces have dimension $mn - 1$)

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Theorem (D-Rodriguez, 2012)

M_r and M_{m-r+1} are ML-dual. □

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Further work

symmetric/alternating matrices!

tensors? other ML-dual pairs of varieties?