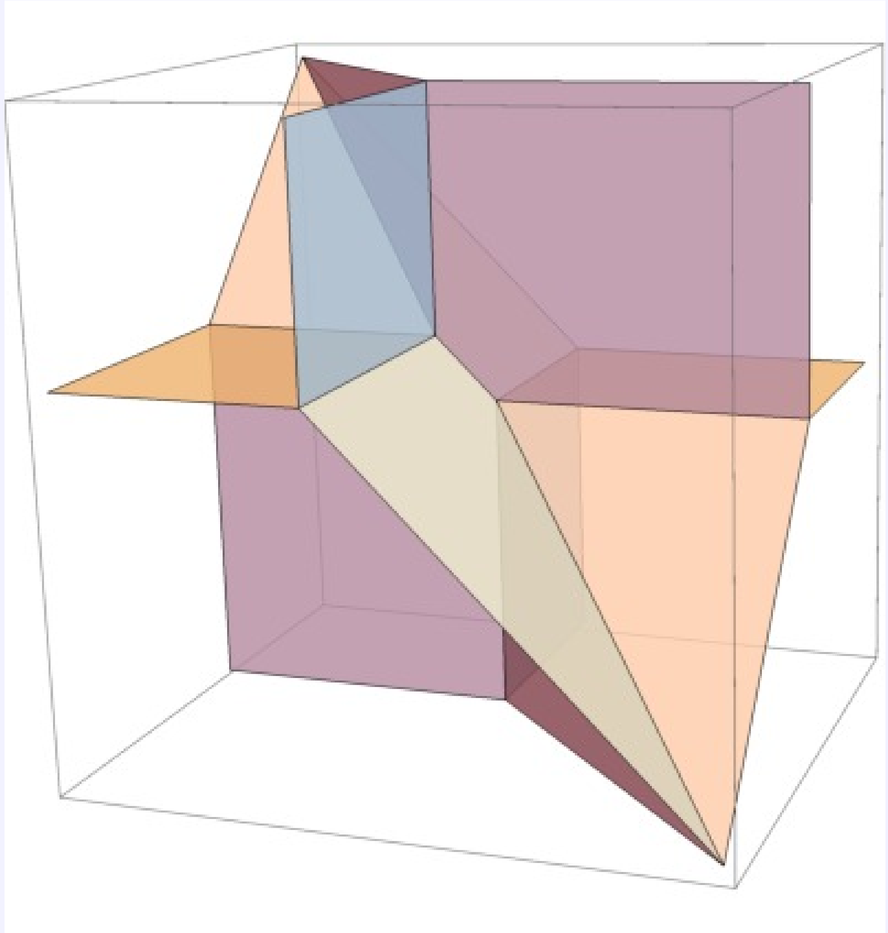
Tropical aspects of algebraic matroids



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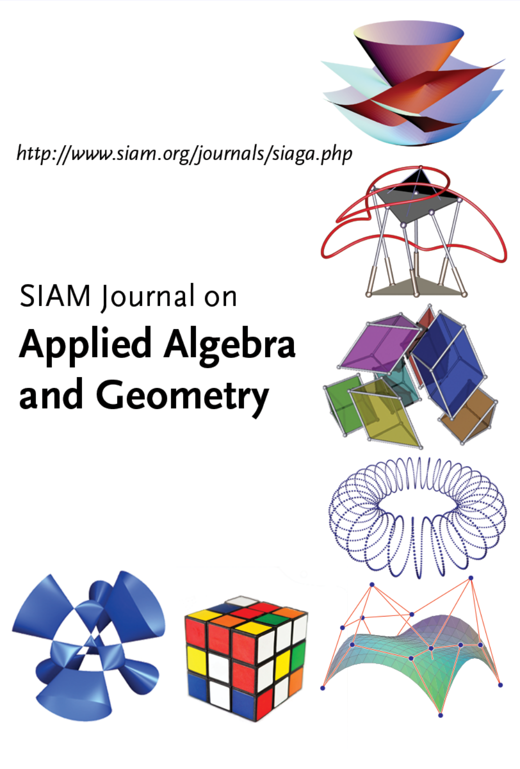
(w/ Rudi Pendavingh and Guus Bollen TU/e)

Two advertisements

SIAGA:



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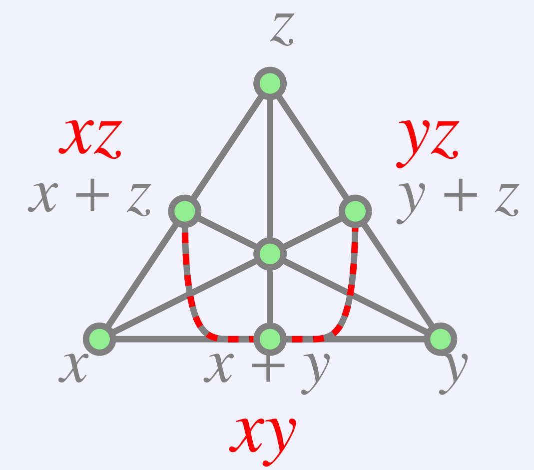
Algebraic matroids



K algebraically closed, $X \subseteq K^E$ irreducible

$\rightsquigarrow$ *algebraic matroid* $M(X)$ on E:
$I \subseteq E$ independent $:\Leftrightarrow X \to K^I$ dominant

Example (Fano and non-Fano): $K = \overline{\mathbb{F}_2}$,
$X = \mathrm{im}[K^3 \to K^7, (x, y, z) \mapsto (x, y, z, \sout{y+z}, \sout{x+z}, \sout{x+y}, \sout{x+y+z})]$

yz xz xy xyz

{algebraic matroids} is closed under deletion and contraction

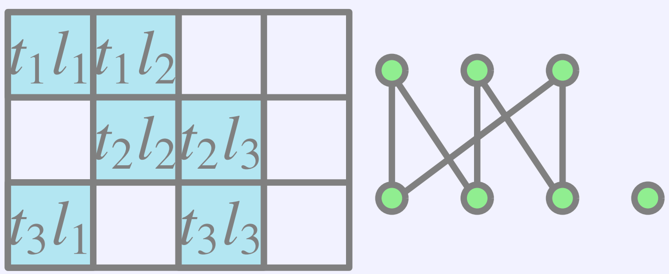
BIG OPEN QUESTIONS:
- is algebraic realisability decidable?
- is the class closed under duality?
- how many algebraic matroids are there?

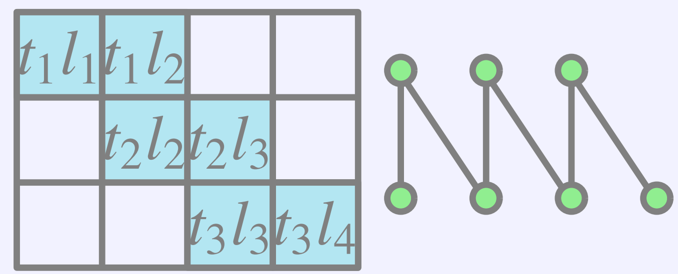
Application: matrix completion

General observation: I independent $\Leftrightarrow$ $\mathrm{Trop}(X) \to \mathbb{R}^I$ surjective.

$X_{m,n,r} = \{\text{matrices of rank} \leq r\} \subseteq K^{m \times n}$
$I \subseteq [m] \times [n]$ independent $\Leftrightarrow$ a general partial I-matrix $/K$ can be completed to a rank-$\leq r$-matrix.

$r = 1$:





Open: can independence in $M(X_{m,n,r})$ be tested in poly time?

Theorem (Daniel Bernstein): For $r = 2$, I is independent iff I has an acyclic orientation without alternating cycles.

Proof uses $\mathrm{Trop}(\mathrm{Gr}_{2,m+n}) \subseteq \mathrm{Trop}(\{\text{skew-symmetric matrices}\})$.

Characteristic zero vs nonzero

Lemma (Ingleton)
If $\mathrm{char} K = 0$, then {matroids algebraic/K}={matroids linear/K}.

Proof: For general $v \in X$, $M(T_v X) = M(X)$.

$\rightsquigarrow$ answers in char 0: yes, yes, 0 percent (Nelson, 2017).

Does not work in char $p > 0$:

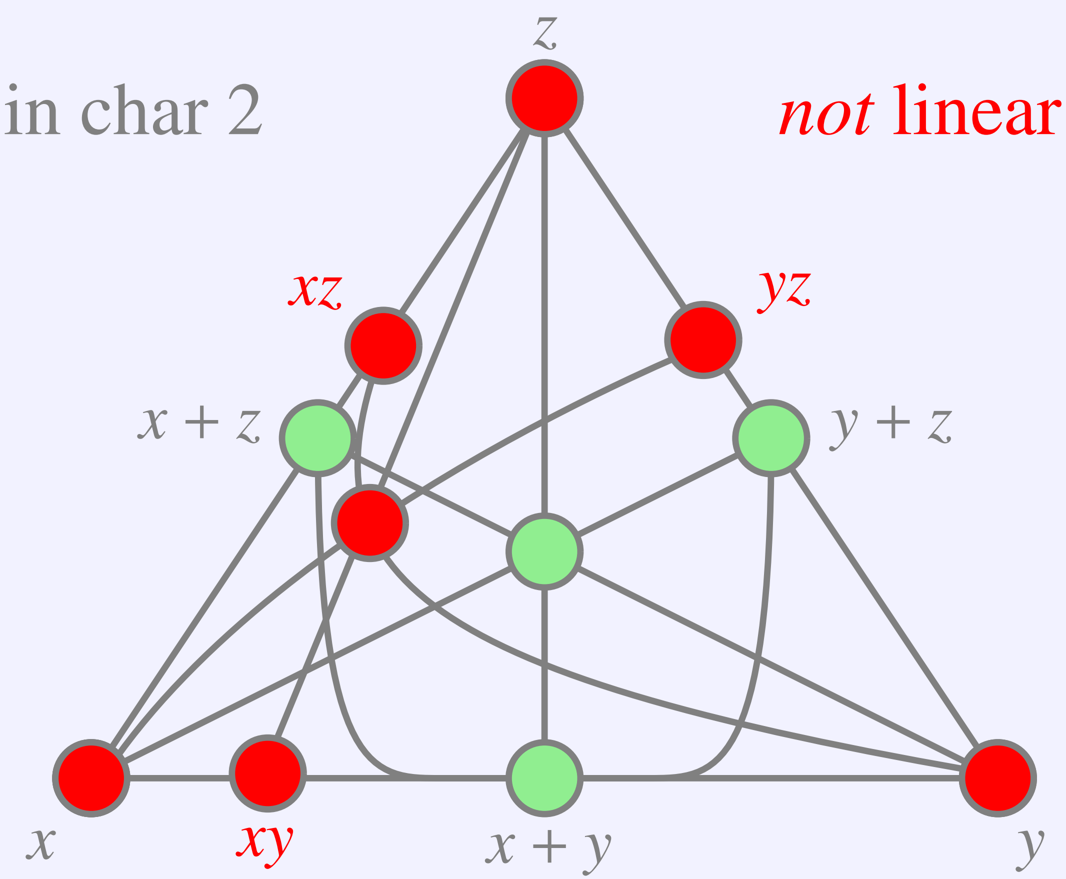
Example
$X = \{(t, t^p) \mid t \in K\}$, $M(X)$ has bases $\{1\}, \{2\}$
but $T_v X = \langle(1, 0)\rangle$ so $M(T_v X)$ only has basis $\{1\}$.

Way out (Lindström): replace X by $\{(x_1^p, x_2) \mid x \in X\} = Y = \{(t, t) \mid t \in K\}$ and $T_v Y = \langle(1, 1)\rangle$ with $M(T_v Y) = M(Y) = M(X)$.

An algebraic but nonlinear matroid

only linear in char 2

not linear in char 2



⇝ $M(X)$ is a nonlinear but algebraic matroid

⇝ cannot find a $Y \subseteq K^{10}$ with $v \in Y$ such that $M(T_v Y) = M(X)$

Way out: *Frobenius flocks!*

$F : a \mapsto a^p$ the Frobenius automorphism
$\mathbb{Z}^E$ acts on K^E via $\alpha v := (F^{-\alpha_i} v_i)_{i \in E}$ by Zariski-homeomorphisms

For $X \subseteq K^E$ and $\alpha \in \mathbb{Z}^E$ have $M(X) = M(\alpha X)$.

Theorem (Bollen–Draisma–Pendavingh)
For general $v \in X$, the map $V : \mathbb{Z}^E \to \mathrm{Gr}(d, K^E)$, $V(\alpha) = T_{\alpha v} \alpha X$ has the following properties:
(FF1) $V_\alpha \cap e_i^\perp = \mathrm{diag}(1, \ldots, 1, 0, 1, \ldots, 1) V_{\alpha + e_i}$
(FF2) $V_{\alpha + \mathbf{1}} = \mathbf{1} V_\alpha$
and moreover $\mathrm{Bases}(M(X)) = \bigcup_\alpha \mathrm{Bases}(M(V(\alpha)))$.

Definition
A map V satisfying (FF1), (FF2) is called a *Frobenius flock*.

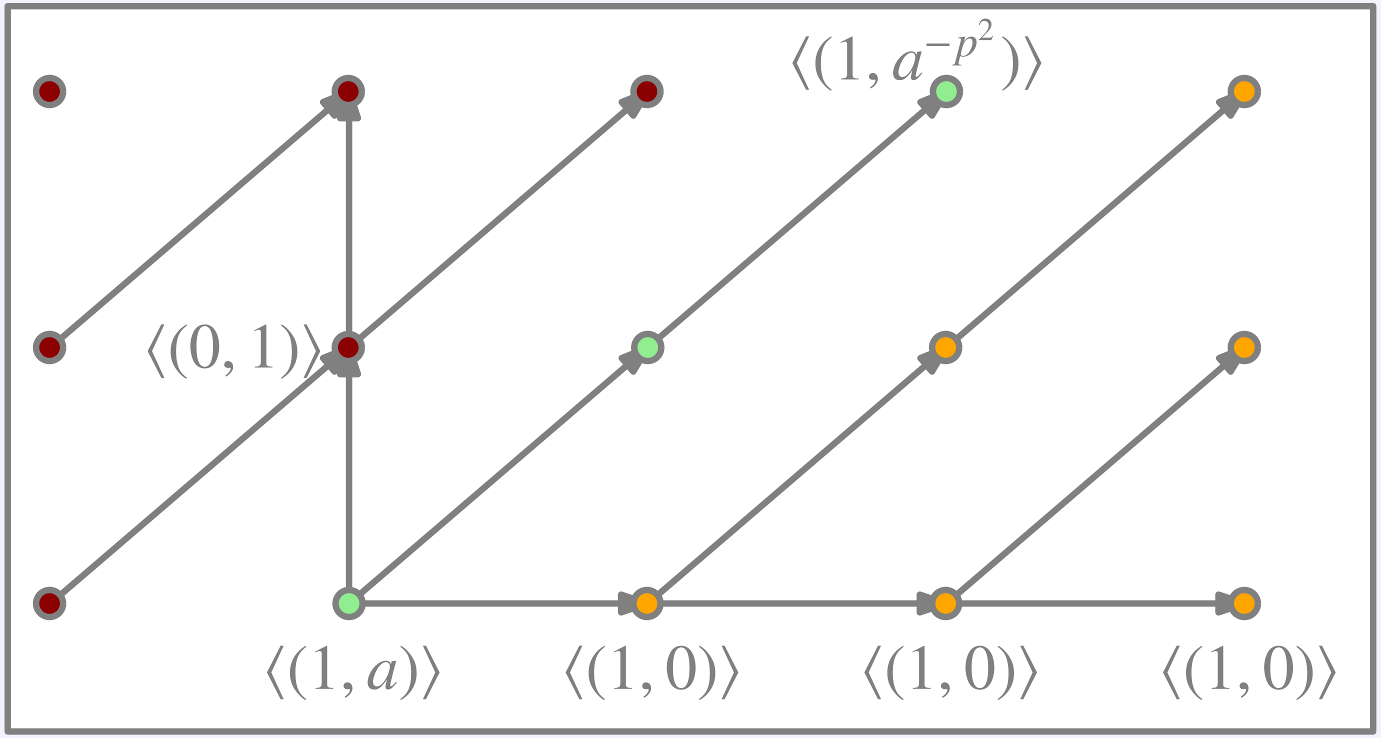
Example 1

(FF1) $V_\alpha \cap e_i^\perp = \mathrm{diag}(1, \ldots, 1, 0, 1, \ldots, 1)V_{\alpha+e_i}$ (FF2) $V_{\alpha+\mathbf{1}} = \mathbf{1}V_\alpha$

$(X =)V_0 = \langle(1, a)\rangle, a \neq 0$

$V_0 \cap e_1^\perp = \{(0,0)\} = \mathrm{diag}(0,1)V_{e_1} \rightsquigarrow V_{e_1} = \langle(1,0)\rangle$

FF2 yields:



Example 2

$$X = \{(x, y, x + y, x + y^{(p^g)}) \mid (x, y) \in K^2\} \subseteq K^4, g > 1, M(X) = U_{2,4}$$

$$T_0X = \text{row space of} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \text{ so } 1, 4 \text{ parallel in } M(T_0X).$$

$$(-e_2 - e_3)X = \{(x, y, x^p + y, x + y^{(p^{g-1})}) \mid (x, y) \in K^2\}$$

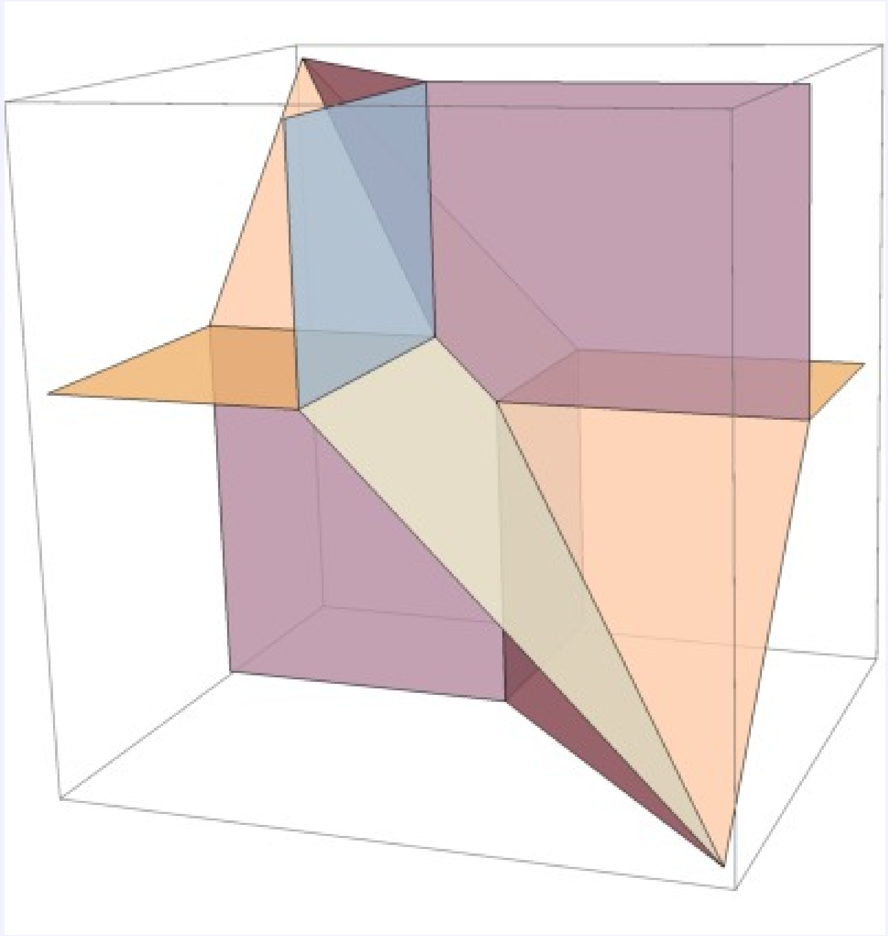
$$T_0(-e_2 - e_3)X = \text{row space of} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; \text{ also } 2, 3 \text{ parallel.}$$

$$(-ge_2 - ge_3)X = \{(x, y, x^{(p^g)} + y, x + y) \mid (x, y) \in K^2\}$$

$$T_0(-ge_2 - ge_3)X = \text{row space of} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}; 1, 4 \text{ indep.}$$

Example 2, continued

Cells where $M(T_{\alpha x}(\alpha X))$ is constant:



These cells are *alcoved polytopes*: max-plus and min-plus closed.

Matroid flocks from valuations and vice versa

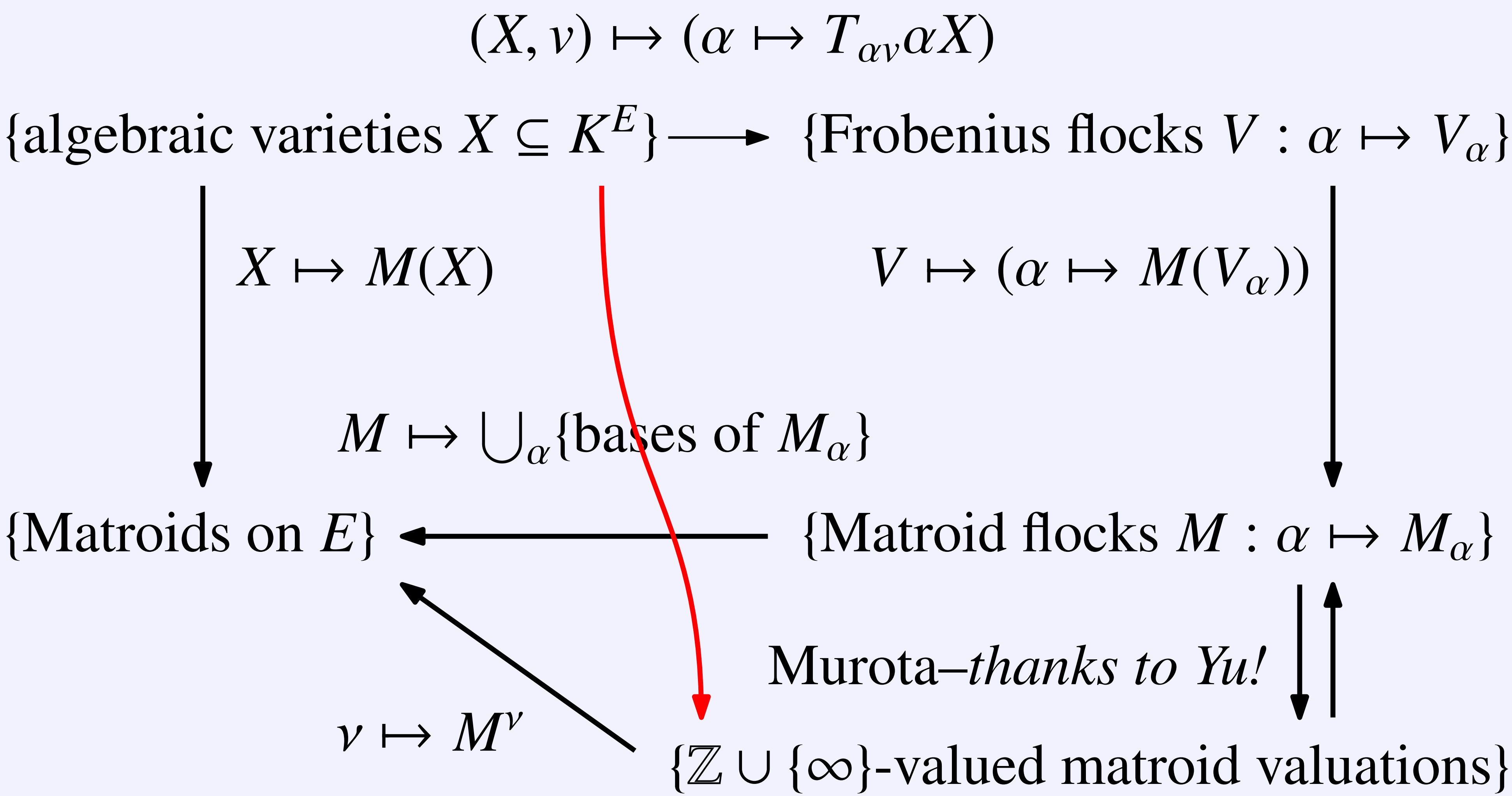
Definition (Dress-Wenzel)
A *matroid valuation* is a map $\nu : \{d\text{-sets in } E\} \to \mathbb{R} \cup \{\infty\}$ such that $\nu(B) \neq \infty$ for some B and $\forall B, B',\ i \in B \setminus B'\ \exists j \in B' \setminus B :$ $\nu(B) + \nu(B') \geq \nu(B - i + j) + \nu(B' + i - j)$.

(ν then lies in the *Dressian* and defines a tropical linear space)

Observations
$\nu \rightsquigarrow$ two matroids: $M^\nu := \{B \mid \nu(B) < \infty\}$ and $\{B \mid \nu(B)$ minimal$\}$; and $\nu'(B) := \nu(B) - \alpha \cdot e_B$ is a valuation for each $\alpha \in \mathbb{R}^E$.

Theorem (Bollen-Draisma-Pendavingh)
Given a $\mathbb{Z} \cup \{\infty\}$-valued ν, set $M^\nu_\alpha := \{B \mid \nu(B) - \alpha \cdot e_B$ minimal$\}$ for each $\alpha \in \mathbb{Z}^E$. This satisfies matroid analogues of FF1,FF2. Conversely, each such *matroid flock* arises in this manner.

Overview



So to a d-dimensional algebraic variety $X \subseteq K^E$ in char p we associate the *Lindstrom valuation* $v^X : \{d\text{-subsets of } E\} \to \mathbb{Z} \cup \{\infty\}$.
Cartwright found a direct construction of v^X.

Monomial parameterisations

$\varphi : (K^*)^d \to (K^*)^n$ monomial map, $\varphi(t) = (t^{Ae_1}, t^{Ae_2}, \ldots, t^{Ae_n})$, where $A \in \mathbb{Z}^{d\times n}$. Set $X := \overline{\mathrm{im}\varphi}$.

Theorem:
ν^X sends $B \subseteq [n], |B| = d$ to the p-adic valuation of $\det A[B]$.

Generalises? G a 1-dimensional algebraic group defined over $\mathbb{F}_p$.
$E := \mathrm{End}(G)$ has $F \in E$.
$A \in E^{d\times n} \rightsquigarrow$ a d-dimensional subgroup $X \subseteq G^n$

Theorem (I think):
$\nu_X(B)$ = number of factors F in the Smith normal form of A.

Definition (Dress-Wenzel)
A matroid M is *rigid* if every valuation ν with $M^\nu = M$ is of the form $M \to \mathbb{R},\ B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{R}^E$.

Theorem
A rigid matroid is algebraically representable over an algebraically closed field K of positive characteristic if and only if it is linearly representable over K.

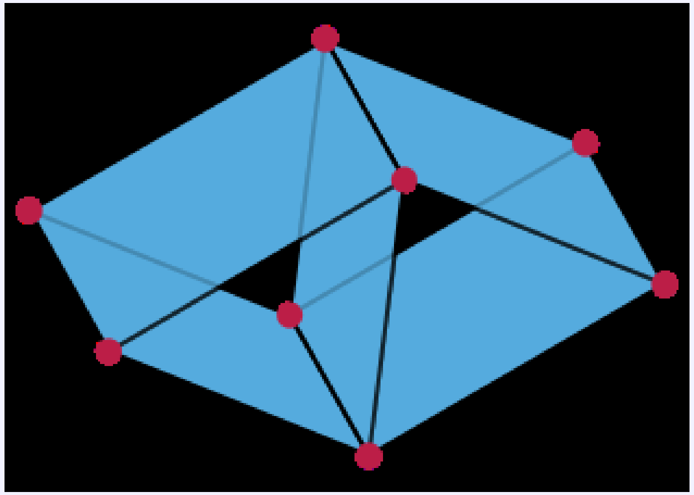
Proof
If X is an algebraic representation, then the Lindström valuation $\nu^X : M(X) \to \mathbb{Z}$ sends $B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{Z}^E$. Then $M^\nu_\alpha = M^\nu$. Now $M(X) = M(T_{\alpha x}\alpha X)$ for $x \in X$ general. □

Applies to projective planes over finite fields!

Properties of Frobenius flocks

Frobenius flocks …

- have deletion/contraction
- are *almost* preserved under duality (replace F by F^{-1})
- allow for circuit hyperplane relaxations
- so Vamos is Frobenius flock realisable (and many more!):



arXiv:1701.06384 (Adv. Math. 323, 2018)

Thank you!