

Finiteness results for infinite-dimensional varieties with many symmetries

Jan Draisma
TU Eindhoven

Institut Mittag-Leffler, 20 January 2011

Fundamental theorem

Increasing maps

$$\text{Inc}(\mathbb{N}) = \{\pi : \mathbb{N} \rightarrow \mathbb{N} \mid \pi(1) < \pi(2) < \dots\}$$

$$\mathbb{C}[x_1, x_2, \dots]$$

$$\pi x_i = x_{\pi(i)}$$

Cohen (1967), Aschenbrenner-Hillar (2007)

$$I_1 \subseteq I_2 \subseteq \dots \text{ ideals in } \mathbb{C}[x_1, x_2, \dots]$$

$\text{Inc}(\mathbb{N})$ -stable

$\Rightarrow$ chain stabilises: $I_n = I_{n+1} = \dots$

“ $\text{Inc}(\mathbb{N})$ -Noetherian”

Cohen (1987), Hillar-Sullivant (2009)

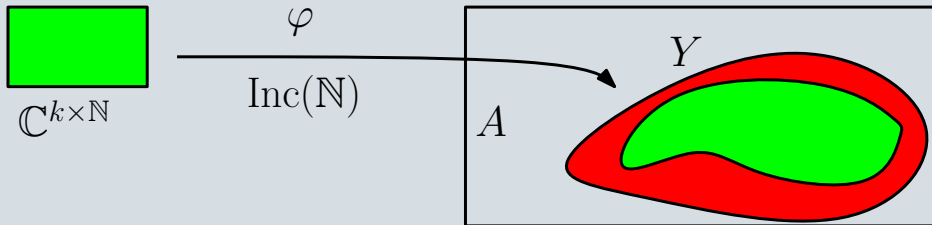
$$\mathbb{C} \begin{bmatrix} x_{11} & x_{12} & \cdots \\ \vdots & \vdots & \\ x_{k1} & x_{k2} & \cdots \end{bmatrix} \text{ is } \text{Inc}(\mathbb{N})\text{-Noetherian.}$$

$\Rightarrow \mathbb{C}^{k \times \mathbb{N}}$ “ $\text{Inc}(\mathbb{N})$ -Noetherian topological space”

Applications

- Independent set theorem (Hillar-Sullivant 2009)
- **Relations among Vandermonde determinants** (Dress 1990s, D 2008)
- Gaussian k -factor model (Drton-Sturmfels-Sullivant 2007, D 2008)
- **Tensors of bounded rank** (Kuttler-D 2010)

Phenomenon



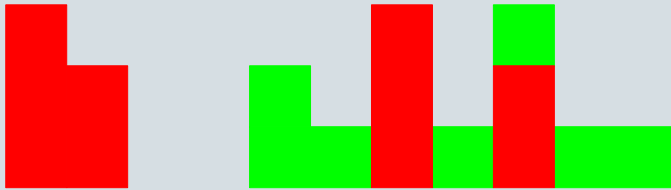
(φ, A) “reasonable” $\Rightarrow \overline{\text{im}(\varphi)}$ defined by finitely many $\text{Inc}(\mathbb{N})$ -orbits of equations.

Proof of fundamental theorem

$\text{Inc}(\mathbb{N})$ -divisibility

u, v monomials in $x_1, x_2, \dots$

$$u|_{\text{Inc}(\mathbb{N})} v \Leftrightarrow \exists \pi \in \text{Inc}(\mathbb{N}) : \pi u|v$$



$$x_1^3 \ x_2^2 |_{\text{Inc}(\mathbb{N})} x_1^2 \ x_2^1 \ x_3^3 \ x_4^1 \ x_5^3 \ x_6^1 \ x_7^1$$

Monomial order

$\leq$ well-order

$$u \leq v \Rightarrow su \leq sv$$

$$u \leq v \Rightarrow \pi u \leq \pi v$$

Exercise

$$u \leq \pi u$$

Proof of fundamental theorem, cont.

Lemma

$u_1, u_2, u_3, \dots$ monomials $\Rightarrow \exists i < j : u_i |_{\text{Inc}(\mathbb{N})} u_j$
("well-quasi-order")

Proof

$u_1, u_2, \dots \leq$ -minimal counter-example ("bad sequence")

$i_1 < i_2 < \dots$ exponent of x_1 increasing weakly

$\Rightarrow u_1, \dots, u_{i_1-1}, u'_{i_1}, u'_{i_2}, \dots$ smaller counter-example

where $u_{i_j} = x_1^{e_{i_j}} \cdot (a \mapsto a + 1)u'_{i_j}$

□

Note

- Lemma $\Rightarrow$ fundamental theorem.
- Finite $\text{Inc}(\mathbb{N})$ -Gröbner basis!

Non-example

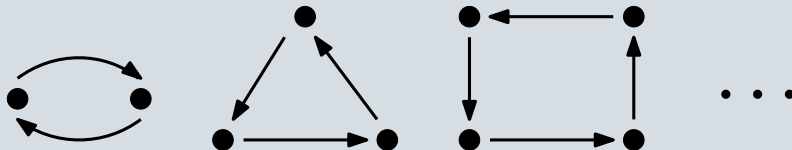
$$\mathbb{C}[y_{ij} \mid i, j \in \mathbb{N}]$$

$$\pi y_{ij} = y_{\pi(i)\pi(j)}$$

not $\text{Inc}(\mathbb{N})$ -Noetherian.

$$I = \langle y_{12}y_{21}, y_{12}y_{23}y_{31}, y_{12}y_{23}y_{34}y_{41}, \dots \rangle_{\text{Inc}(\mathbb{N})}$$

not finitely generated.



Relations among Vandermonde

$$k, n \in \mathbb{N}$$

$$v_S := \prod_{i,j \in S, i > j} (x_i - x_j) \in \mathbb{C}[x_1, \dots, x_n], \quad S \in \binom{[n]}{k}$$

Dress (1990s)

Polynomial relations among v_S stabilise for $n \rightarrow \infty$?

D (2009)

Yes!

Recall

$$S = \{i_1 < \dots < i_k\}$$

$$v_S = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_{i_1} & x_{i_2} & \dots & x_{i_k} \\ x_{i_1}^2 & x_{i_2}^2 & \dots & x_{i_k}^2 \\ \vdots & \vdots & & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \dots & x_{i_k}^{k-1} \end{bmatrix}$$

Relations among Vandermonde, cont.

Proof

Pass to $n = \infty$.

$(\varphi : \mathbb{C}^{\mathbb{N}} \rightarrow \mathbb{C}^{\binom{\mathbb{N}}{k}} =: A \text{ with coordinates } y_S)$

$\mathbb{C}[v_S | S \in \binom{\mathbb{N}}{k}]$ is quotient of

$R = \mathbb{C}[y_S | S \in \binom{\mathbb{N}}{k}] / \text{Plücker relations}$
(finitely many up to $\text{Inc}(\mathbb{N})$; define Y).

$R = \mathbb{C}[z_{ij} | i \in [k], j \in \mathbb{N}]^{\text{SU}_k}$

Reynolds: $\rho : \mathbb{C}[z_{ij} | i, j] \mapsto R, f \mapsto \int_{\text{SU}_k} g f dg$

$I_1 \subseteq I_2 \subseteq \dots$ $\text{Inc}(\mathbb{N})$ -stable in R

$J_i = \mathbb{C}[z_{ij} | i, j] \cdot I_i$

fundamental theorem: $J_n = J_{n+1} = \dots$

$\Rightarrow I_n = \rho(J_n) = \rho(J_{n+1}) = I_{n+1} = \dots$

□

Proposition (Krasilnikov, 2010)

$\mathbb{F}_2[x_i y_j - x_j y_i \mid i, j \in \mathbb{N}]$ is *not* $\text{Inc}(\mathbb{N})$ -Noetherian!

Tensors of bounded rank

$V_1, \dots, V_p$ vector spaces

$V_1 \otimes \dots \otimes V_p$ tensor product

Rank ≤ 1

$$\mu = v_1 \otimes \dots \otimes v_p$$

Rank $\leq k$

$$\omega = \mu_1 + \dots + \mu_k, \mu_i \text{ rank} \leq 1$$

Border rank $\leq k$

Zariski closure

Theorem (D-Kuttler, 2010)

$$\forall k \exists d \forall p \forall V_1, \dots, V_p:$$

$\{\text{tensors of border rank} \leq k \text{ in } V_1 \otimes \dots \otimes V_p\}$

defined set-theoretically in degree $\leq d$.

Why border rank?

Example

$$\omega \in \mathbb{C}^{2 \times 2 \times 2}$$

$$\psi : \mathbb{C}^2 \rightarrow \mathbb{C}^{2 \times 2},$$

$$x \mapsto \begin{bmatrix} x_1\omega(1, 1, 1) + x_2\omega(1, 1, 2) & x_1\omega(1, 2, 1) + x_2\omega(1, 2, 2) \\ x_1\omega(2, 1, 1) + x_2\omega(2, 1, 2) & x_1\omega(2, 2, 1) + x_2\omega(2, 2, 2) \end{bmatrix}.$$

Cases

1. $\dim \operatorname{im} \psi = 0, 1$: $\operatorname{rk} \omega \leq 2$
2. $\operatorname{im} \psi = \langle A, B \rangle$ with $\operatorname{rk}(A) = \operatorname{rk}(B) = 1$: $\operatorname{rk} \omega = 2$
3. otherwise: ω has rank 3

Case 3: ω zero of discriminant($\det \psi(x_1, x_2)$)

Cayley's hyperdeterminant

$\rightsquigarrow \{\omega \mid \operatorname{rk} \omega \leq 2\}$ is complicated;

$$\{\omega \mid \text{border rank} \leq 2\} = \mathbb{C}^{2 \times 2 \times 2}$$

Border rank ≤ 2

Flattening

$$\flat : V_1 \otimes \cdots \otimes V_p \rightarrow (V_1 \otimes \cdots \otimes V_q) \otimes (V_{q+1} \otimes \cdots \otimes V_p)$$

does not increase rank.

Theorem (Landsberg-Manivel 2004)

3×3 -minors of flattenings

define “border rank ≤ 2 ”.

Conjecture (Garcia-Stilman-Sturmfels 2005)

Theorem (Raicu 2010)

Actually, generate their ideal.

Tensors of bounded rank, proof

May take all V_i equal to V of fixed dimension $n \geq k + 1$
(Allman-Rhodes, Landsberg-Manivel-Weyman).

Pass to $p = \infty \rightsquigarrow A = \lim_{\leftarrow p} V^{\otimes p}$

$X^{\leq k} = \{\text{border rank} \leq k\} \subseteq A$.

$\Pi = \bigcup_p \text{Sym}(p) \ltimes \text{GL}(V)^p$ natural symmetries

$Y^{\leq k}$ defined by $(k + 1) \times (k + 1)$ -minors

(finitely many Π -orbits suffice)

$X^{\leq k} \subseteq Y^{\leq k}$

Claim:

$Y^{\leq k}$ is Π -Noetherian topological space

induction on k , uses fundamental theorem in each step.

$\rightsquigarrow X^{\leq k} \subseteq Y^{\leq k}$ defined by finitely many Π -orbits of equations.

□

Further issues

1. Buchberger algorithm

(Cohen 1987, La Scala-Levandovskyy 2010, Brouwer-D 2010)

(2. Application to GSS conjecture?)

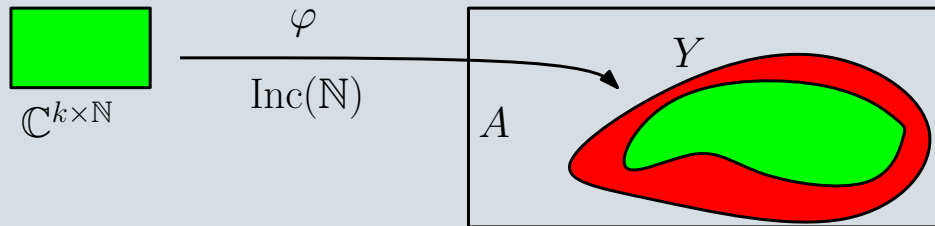
3. Other monoids and well-quasi-orders?

(Higman!, Kruskal?, Robertson-Seymour??)

4. Over real numbers, Euclidean closure?

5. What is “reasonable” in the

Phenomenon?



(A, φ) “reasonable” $\Rightarrow$ image defined by finitely many $\text{Inc}(\mathbb{N})$ -orbits of equations.