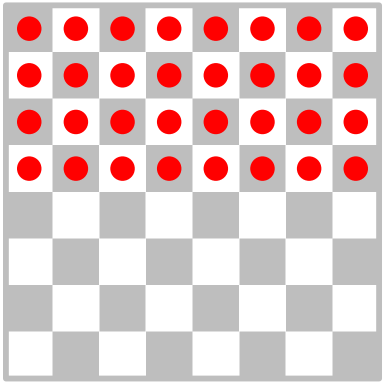
Energy minimisation on a toric grid

Jan Draisma
Eindhoven University of Technology

3 May 2012, Kolloquium Mainz

relax and diagonalise

Repelling particles



minimal-energy configuration?

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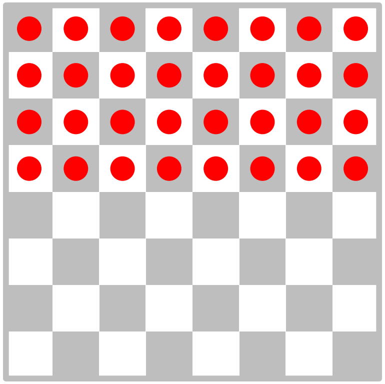

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Main result

(Bouman-D-v. Leeuwaarden)

m, n multiples of 4
$m \times n$ toric chessboard
$mn/2$ repelling particles

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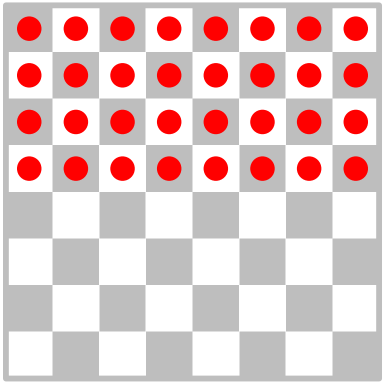

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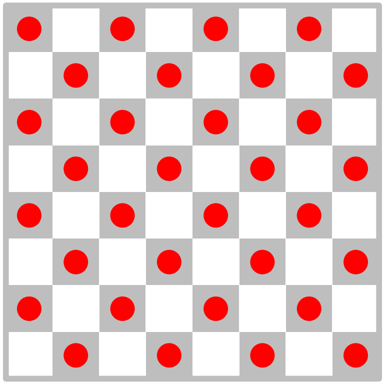

Checkerboard conjecture

m, n *even* suffices

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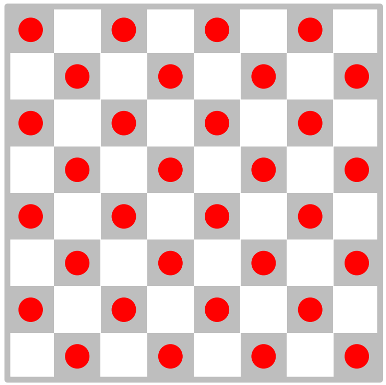

Checkerboard conjecture

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Energy

(X, d) compact metric space

$f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$
decreasing force

$P \subseteq X$ has *energy*
$E(P) := \sum_{x,y \in P,\ x \neq y} f(\delta(x, y))$

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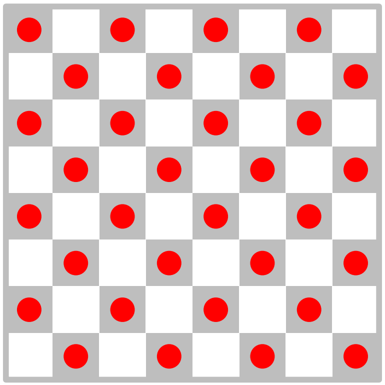

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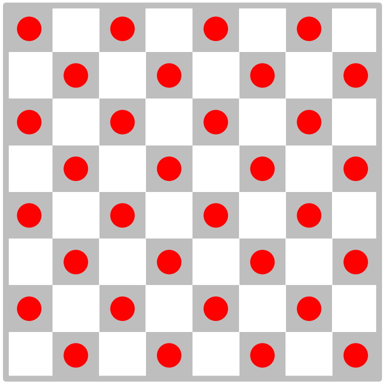

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Complete monotonicity

means $(-1)^k f^{(k)} > 0$
for all $k = 0, 1, 2, \ldots$

examples:
$f(s) = s^{-a}$ with $a > 0$
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(Cohn-Kumar)

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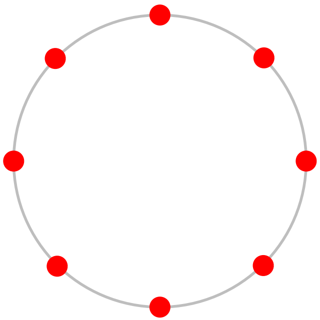
for *all* completely monotonic f

Universal optima on spheres

(Cohn-Kumar)
regular polygons



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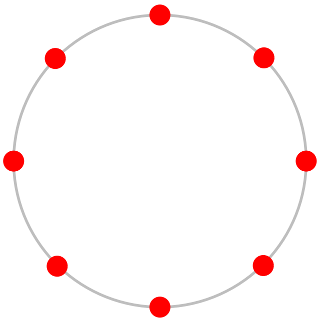
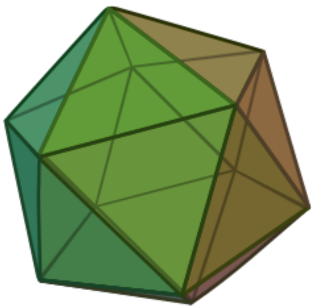


icosahedron



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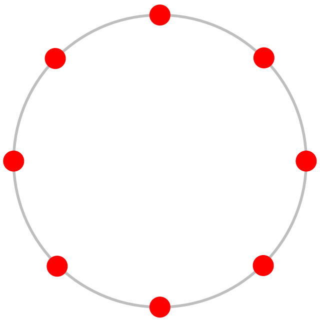

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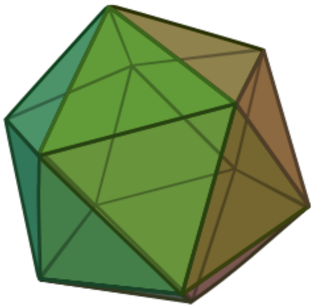
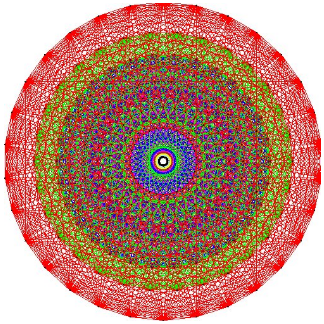


(image from Wikipedia)

root vectors of E_8



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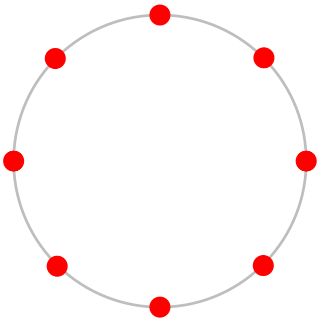

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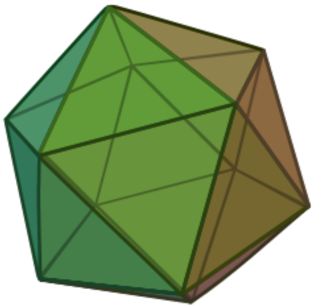

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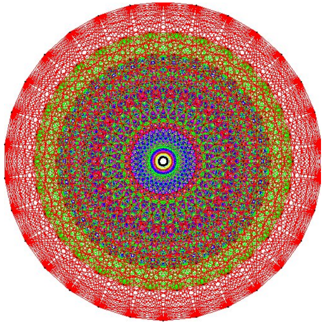

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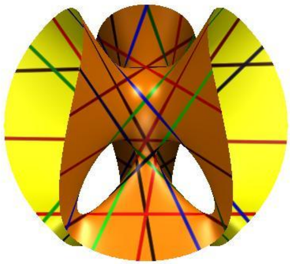
Schläfli configuration on S^5

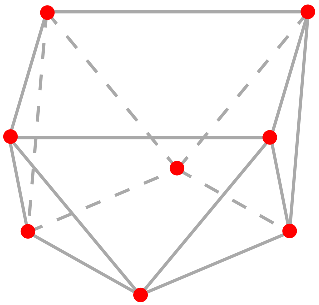


(image from Mainz)

relax and **diagonalise**

Universal optima are rare

non-examples:
five points on S^2
dodecahedron
cube



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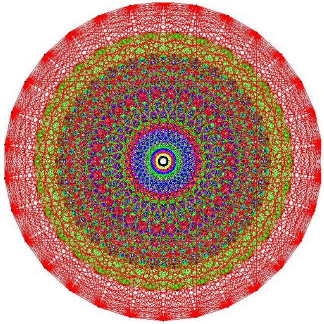

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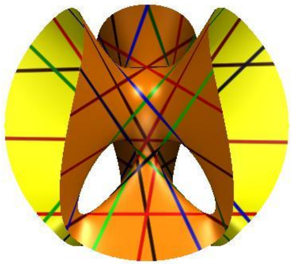

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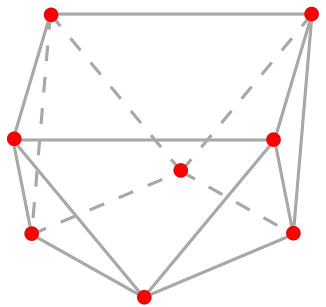

Theorem

(Bouman-D-Leeuwaarden)
$n_1, \ldots, n_d \in \mathbb{N}$
$4|n$ or $n_i = 2$
$G := (\mathbb{Z}/n_1) \times \ldots \times (\mathbb{Z}/n_d)$
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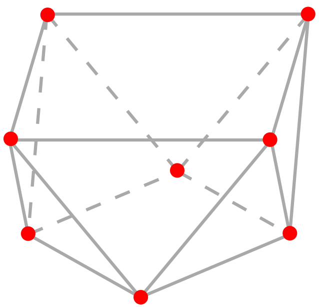

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$P_{0/1} := \{(i_1, \ldots, i_d) \mid$
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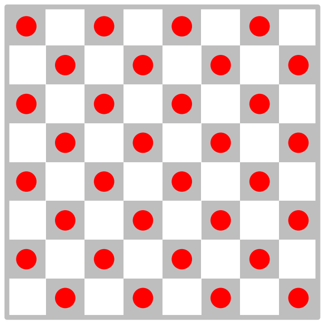
are universally optimal



relax and diagonalise

Relaxation

$p := n_1 \cdots n_d/2$
$V := \mathbb{R}^G$
$(.|.)$ inner product

$P \subseteq G, |P| = p$
$x_P \in V$ characteristic vector
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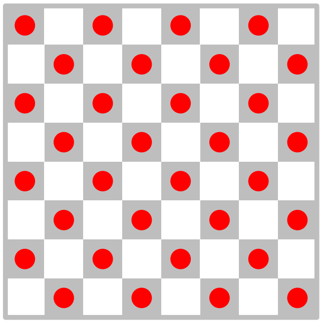

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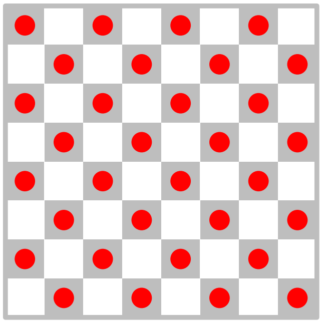

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G acts on V
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$= E(P)$

Fractional energy minimisation

minimise $(x|Ax)$
subject to
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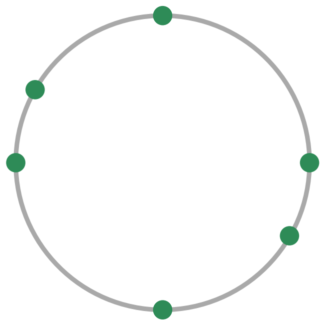

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$$
\begin{aligned}
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&= \sum_{h,k \in P, k \neq h} f(\delta(h,k)) \\
&= E(P)
\end{aligned}
$$

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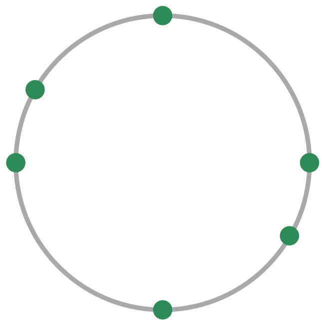

Eigenspaces of A

A symmetric
$\mathbf{1}$ eigenvector
so $x = \mathbf{1}/2 + y$ with $y \perp \mathbf{1}$
$(\mathbf{1}/2|\mathbf{1}/2) = |G|/4$

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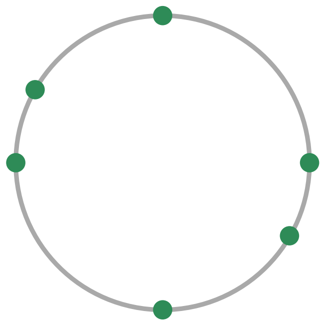

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minimise $(y|Ay)$
subject to
$y \perp \mathbf{1}$
$(y|y) = |G|/4$

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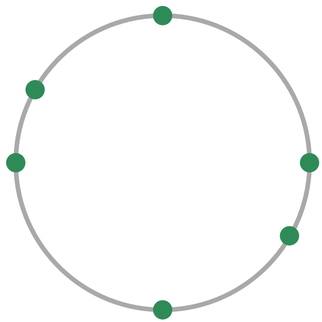

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$y \perp \mathbf{1}$
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take y in eigenspace
with *smallest eigenvalue*!

Diagonalising A

$V = \mathbb{R}^G$ regular representation
$V_{\mathbb{C}} = \bigoplus_\chi V_\chi$
diagonalises each $g \in G$

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$(\mathbf{1}/2|\mathbf{1}/2) = |G|/4$

minimise $(y|Ay)$
subject to
$y \perp \mathbf{1}$
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take y in eigenspace
with *smallest eigenvalue*!

Diagonalising A

$V = \mathbb{R}^G$ regular representation
$V_{\mathbb{C}} = \bigoplus_\chi V_\chi$
diagonalises each $g \in G$

$A = \sum_{g \neq 0} f(\delta(g, 0))g \in \mathbb{R}G$
$\lambda(\chi) = \sum_{g \neq 0} f(\delta(g, 0))\chi(g)$
eigenvalue of A on V_χ

Eigenspaces of A

A symmetric
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$\mathbf{1}/2 + y = x_{P_{0/1}}$

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Two-dimensional grid

$m := n_1$, $n := n_2$

$\zeta, \omega \in \mathbb{C}^*$
$\zeta^m = \omega^n = 1$
$\lambda(\zeta, \omega)$
$= \sum_{(i,j) \neq (0,0)} f(\delta((i, j), (0, 0)))\zeta^i \omega^j$

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minimum at $(\zeta, \omega) = (-1, -1)$??
would give

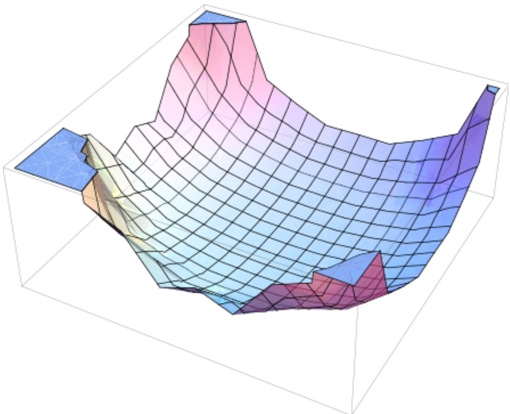
$$y = \frac{1}{2}\begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix}$$

so that $\mathbf{1}/2 \pm y = x_{P_{0/1}}$

Fourier transform

$m = n = 10$
$f(s) = \frac{1}{s}$

graph of $\lambda(\zeta, \omega)$



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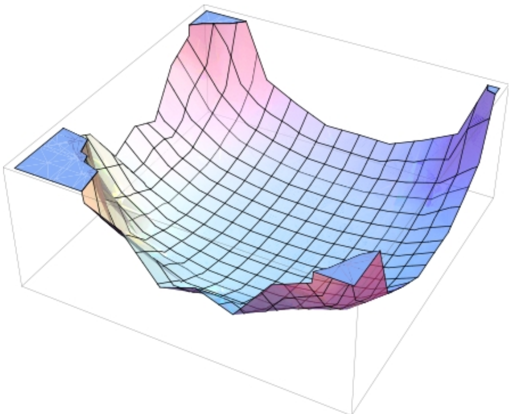

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$\delta((i,j),(0,0)) = \delta(i,0) + \delta(j,0)$

$\delta(i,0) = |i|$ if $-m/2 \le i \le m/2$

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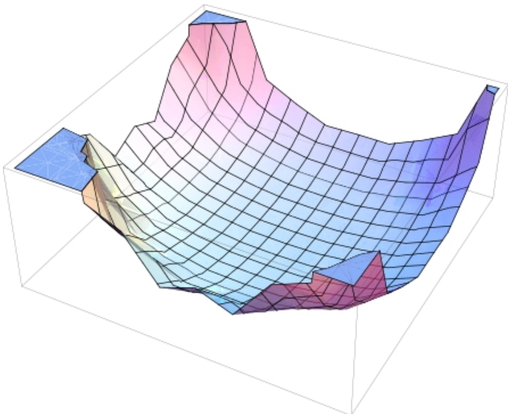

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$f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$
completely monotonic

$\rightsquigarrow f(s) = \int_0^\infty e^{-ts} \mathrm{d}\alpha(t)$
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so optimality of
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for $f(s) = a^{-s}$, $a > 1$
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Universal optimality proof

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$\lambda(\zeta)$
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$+ a^{-m/2}\zeta^{m/2}$
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projective spaces

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relax and diagonalise!