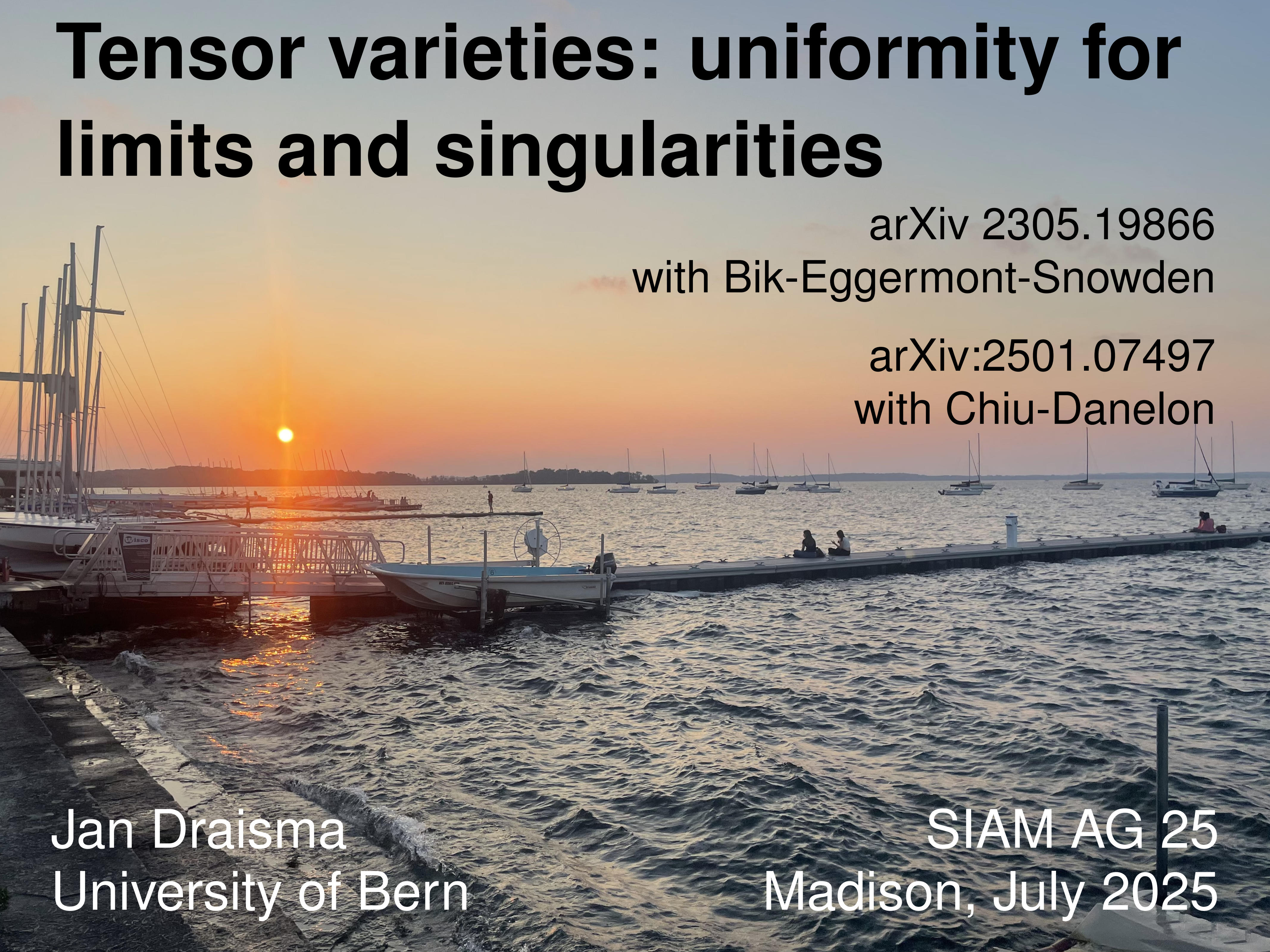
Tensor varieties: uniformity for limits and singularities

arXiv 2305.19866
with Bik-Eggermont-Snowden

arXiv:2501.07497
with Chiu-Danelon



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Example (border quadric rank ≤ 3)

$$Y(V) = \overline{\{f_1 g_1 + f_2 g_2 + f_3 g_3 \mid f_i, g_i \in S^2 V\}} \subseteq S^4 V.$$

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Morphism $\alpha : X \to Y$:

$V \mapsto \alpha_V : X(V) \to Y(V)$ s.t. $\forall \varphi$:

$$\begin{array}{ccc} X(V) & \xrightarrow{\alpha_V} & Y(V) \\ \downarrow X(\varphi) & \circlearrowleft & \downarrow Y(\varphi) \\ X(W) & \xrightarrow{\alpha_W} & Y(W) \end{array}$$

Image closure of morphisms

$\alpha : X \to Y$ a morphism $\rightsquigarrow$ $\overline{\mathrm{im}(\alpha)} : V \mapsto \overline{\mathrm{im}(\alpha_V)}$ is a tensor subvariety of Y.

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Example [Ballico-Bik-Oneto-Ventura, 2022]

$X(V) = (S^2 V)^6$ (six quadrics), $Y(V) = S^4 V$ (one quartic)

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But $\overline{\text{im}(\alpha_V)}$ also contains $\lim_{\epsilon\to 0}$
$\frac{1}{\epsilon}[(x^2+\epsilon g)(y^2+\epsilon f) - (u^2-\epsilon q)(v^2-\epsilon p) - (xy+uv)(xy-uv)] =$
$x^2f + y^2g + u^2p + v^2q$.

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Theorem [BBOV]: $\mathrm{im}(\alpha_V)$ is not closed for $\dim(V) \gg 0$.

Uniformity for limits

Theorem 1 [Bik-D-Eggermont-Snowden, 2023]
$\alpha : X \to Y$ a morphism, then $\exists N \in \mathbb{N}$ such that for all V and all $y \in \overline{\mathrm{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

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Application to de-bordering

$\beta : T_1 \to T_2$ morphism between tensor spaces with $\mathrm{im}(\beta_V)$ a cone spanning $T_2(V)$. For $f \in T_2(V)$ define
$R(f) := \min\{r \mid f = \sum_{i=1}^{r} \beta_V(h_i)\}$ (*rank*) and
$\underline{R}(f) := \min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} \beta_V(h_i(\epsilon))\}$ (*border rank*).

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Corollary
There is a function $F : \mathbb{N} \to \mathbb{N}$ such that for all V and $f \in T_2(V) : R(f) \leq F(\underline{R}(f))$.

Uniformity for singular loci

Example (determinantal variety)
$T(V) = V \otimes V$, $X(V) = \{A \mid \mathrm{rk}(A) \leq 3\} \rightsquigarrow$
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The singular locus is again a tensor variety.

Uniformity for singular loci

Theorem 2 [Chiu-Danelon-D, 2024]

X a tensor variety, then $\exists$ closed tensor subvariety $X^{\text{sing}} \subseteq X$ s.t. $X^{\text{sing}}(V) = \text{Sing}(X(V))$ for all V with $\dim(V) \gg 0$.

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Moreover, if $X \subseteq T$, then $\exists U$ s.t. for all V the scheme $Y := \bigcap_{\varphi: V \to U} T(\varphi)^{-1}(X(U))$ satisfies $Y^{\text{red}} = X(V)$ and Y is reduced at all $p \in X(V) \setminus X^{\text{sing}}(V)$.

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In English: *the equations for $X(U)$ pull back to equations that define $X(V)$ as a set, and that define a reduced scheme outside* $\text{Sing}(X(V))$*.*

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Theorem 1
$\alpha : X \to Y$ a morphism, then $\exists N \in \mathbb{N}$ such that for all V and all $y \in \overline{\text{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

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Thank you!