Technische Universiteit
Eindhoven
University of Technology

Min-plus products of distance matrices

(or: The lossy gossip monoid)

Jan Draisma

April 2012
ICMS Edinburgh

joint work with Bart Frenk

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The car-bike metric

Ehv-border A'dam-center A'dam

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 6.8 & 7 \\ 6.8 & 0 & 0.5 \\ 7 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & \color{red}{2.5} \\ 2 & 0 & 0.5 \\ \color{red}{3} & 0.5 & 0 \end{bmatrix}$$

Structure theorem

$D_n := \{(d_{ij})_{ij} \in (\mathbb{R} \cup \{\infty\})^{n\times n} \mid$
$d_{ii} = 0, d_{ij} = d_{ji}, d_{ij} + d_{jk} \geq d_{ik}\}$
cone of *distance matrices*

$M_n := \{d_1 \odot \cdots \odot d_k \mid$
$k \in \mathbb{N}, d_1, \ldots, d_k \in D\}$
monoid generated by D_n

Theorem
M_n is the support of a finite polyhedral fan of dimension $\binom{n}{2}$.
For $n = 2, 3, 4(?)$ it is pure and connected in codimension 1.

$M_2 = D_2$:

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \oplus b \\ a \oplus b & 0 \end{bmatrix}$$

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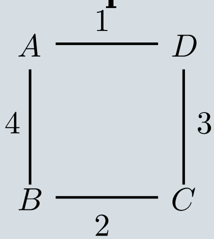
The gossip problem

n people with n secret gossips shared through phone calls

Classical question
minimal number of phone calls before everyone knows all gossips?

Answer
$2n - 4$ for $n \geq 4$
(Baker-Shostak, Bumby, Hajnal-Milner-Szemerédi, Tijdeman)

Example for $n = 4$



Uncertainty matrices

define $U = (u_{ij})_{ij}$

$$u_{ij} := \begin{cases} 0 & i \text{ knows gossip } j \\ \infty & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} \rightsquigarrow$$

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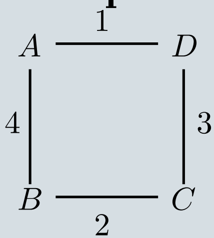

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The gossip monoid

$(2,4)$ call = *left multiplication* with

$$C_{24} := \begin{bmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & 0 \\ \infty & \infty & 0 & \infty \\ \infty & 0 & \infty & 0 \end{bmatrix}$$

Definition
$G_n :=$ monoid generated by
$C_{ij}, 1 \leq i < j \leq n$
the *gossip monoid*

Open questions
-maximal *length* of an element?
-cardinality of G_n?
-efficient membership test?

Observation
C_{ik} distance matrix
$\rightsquigarrow G_n \subseteq M_n$

Some numbers

Jochem Berndsen (TU/e):

n	$\lvert G_n \rvert$	max length
1	1	0
2	2	1
3	11	3
4	189	4
5	9152	6
6	1,092,473	10
7	293,656,554	13
8	166,244,338,221	17

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Lossy gossip

i knows fraction p_{ij} of gossip j

(i, k) **lossy phone calls:**
-fraction q transmitted correctly
-fraction qp_{ij} of gossip j received correctly by k
-accepted if $qp_{ij} > p_{kj}$
-and vice versa

The lossy gossip monoid

define $u_{ij} := -\log p_{ij} \in [0, \infty]$
$U = (u_{ij})_{ij}$ *uncertainty matrix*
$a := -\log q$ *channel unreliability*

Example
$(2, 4)$ lossy phone call =
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$$C_{24}(a) := \begin{bmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & a \\ \infty & \infty & 0 & \infty \\ \infty & a & \infty & 0 \end{bmatrix}$$

Lemma
M_n is the monoid generated by all lossy phone calls $C_{ij}(a),\ 1 \leq i < j \leq n,\ a \in [0, \infty]$.

($\subseteq$: $D = \bigodot_{i<j} C_{ij}(d_{ij})$ for $D \in D_n$)

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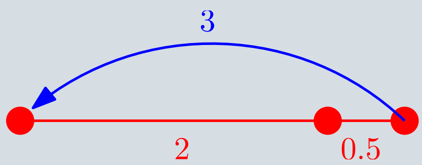
Three persons

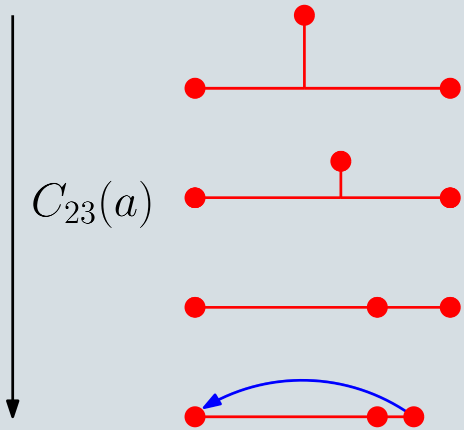
$$\begin{bmatrix} 0 & 2 & 2.5 \\ 2 & 0 & 0.5 \\ 3 & 0.5 & 0 \end{bmatrix} = C_{12}(2) \odot C_{23}(.5) \odot C_{13}(3)$$

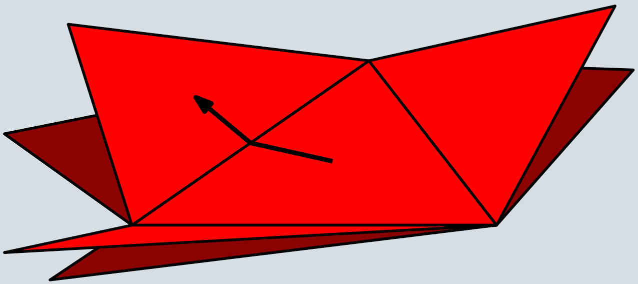


six 3-dimensional cones of type

$$\left\{ \begin{bmatrix} 0 & a & a+b \\ a & 0 & b \\ c & b & 0 \end{bmatrix} \mid c \geq a+b \right\}$$

Three persons, continued





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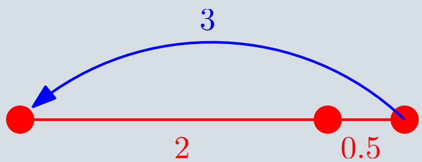

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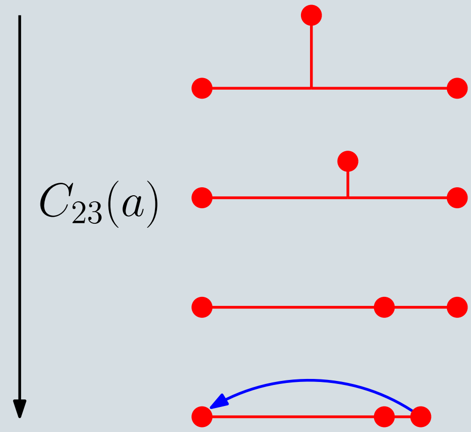
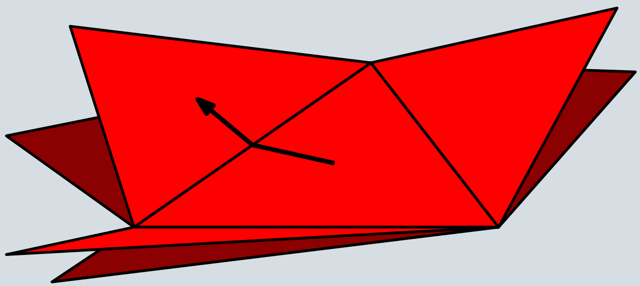




Maps

gossip to lossy gossip
$G_n \longrightarrow M_n$ monoid embedding

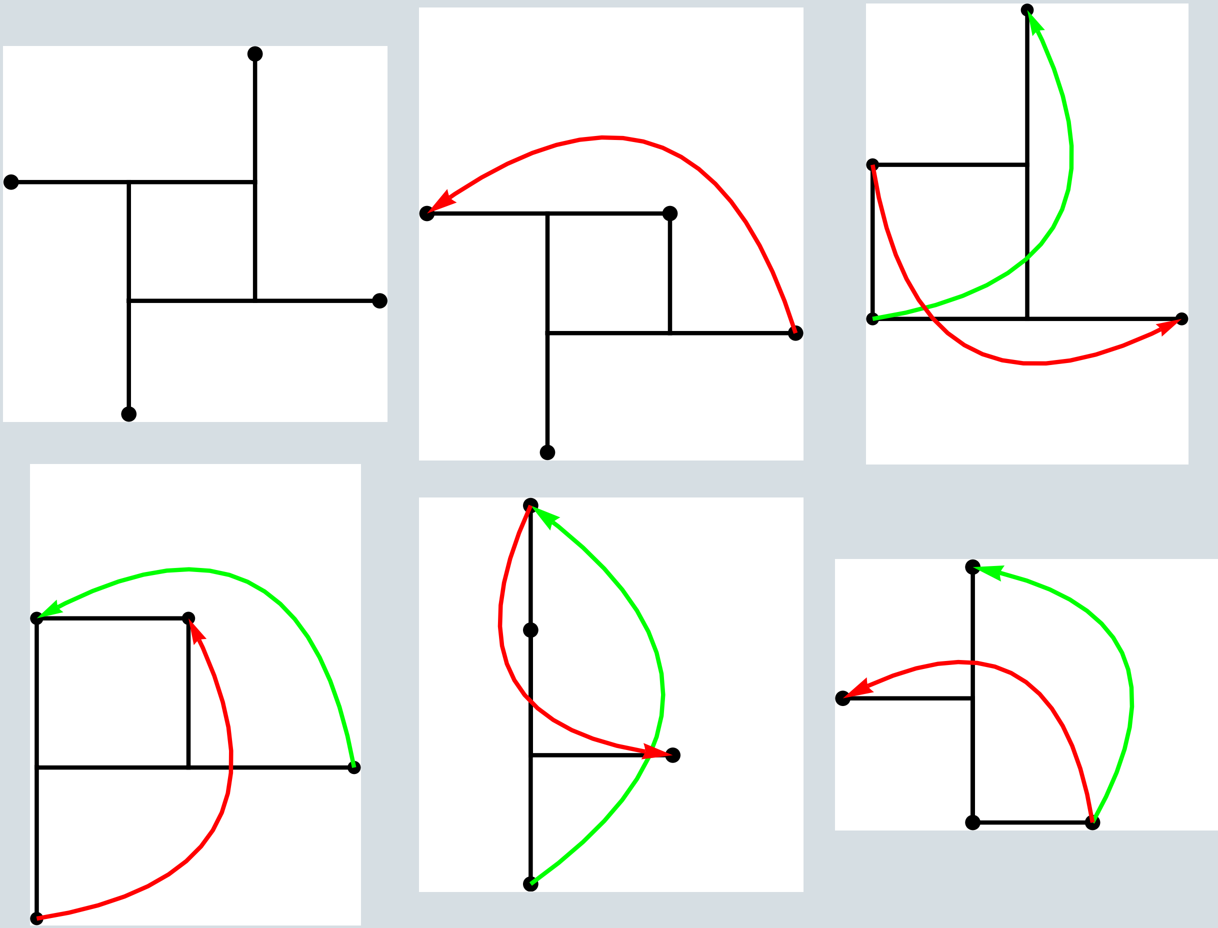
lossy gossip to gossip
$M_n \longrightarrow G_n$ monoid surjection
$\mathbb{R} \mapsto 0, \ \infty \mapsto \infty$

distance to lossy gossip
$D_n \longrightarrow M_n$ subset

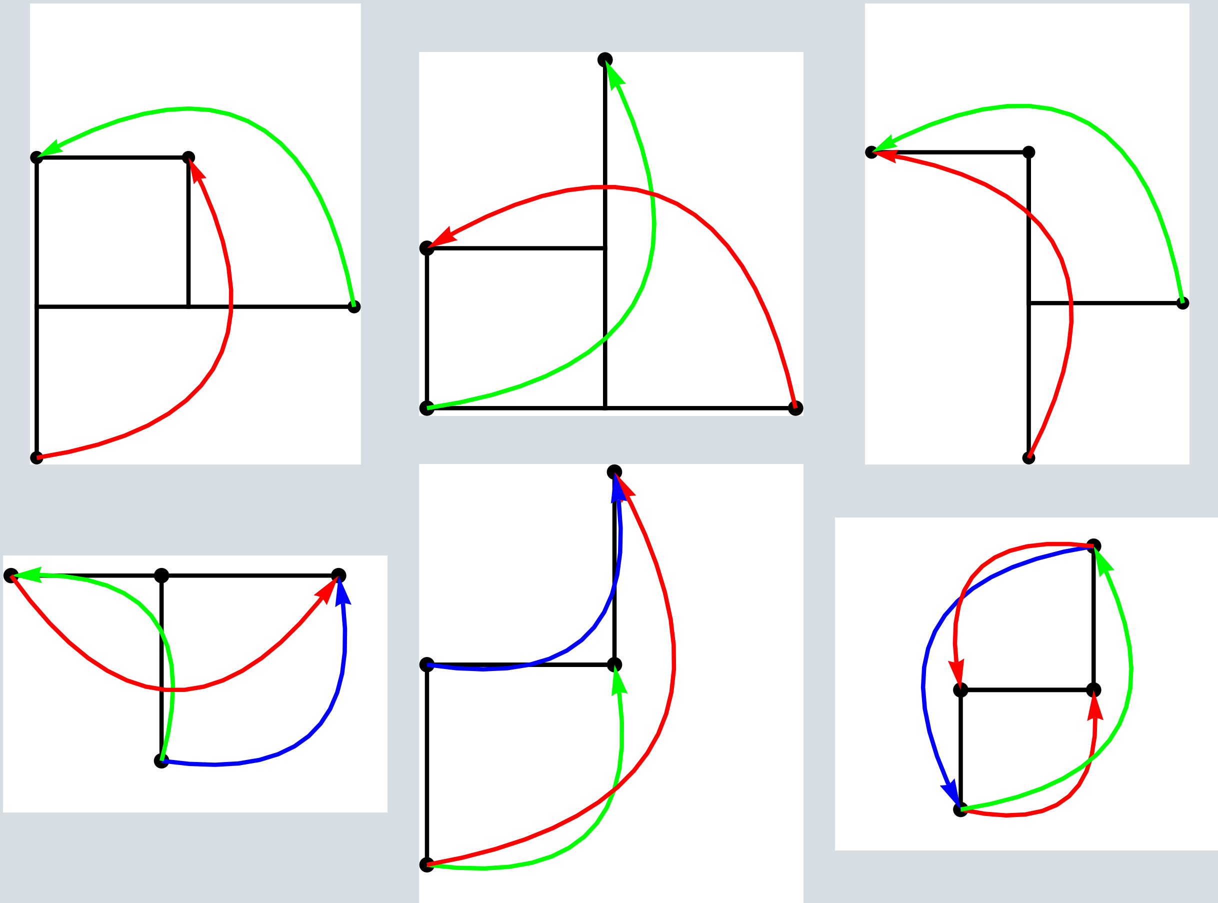
lossy gossip to distance
$M_n \longrightarrow D_n, \ A \mapsto A^{\odot n}$
underlying metric

Visualising
-draw underlying metric
-draw arrows for *detours*

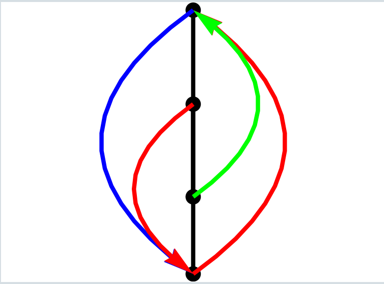
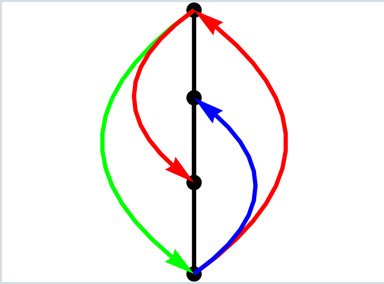
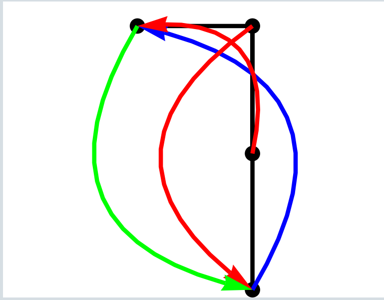
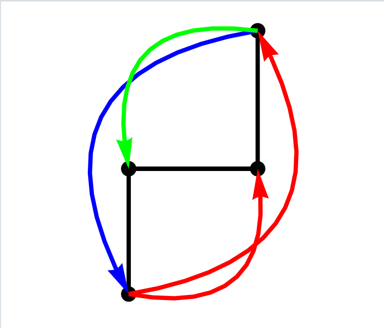
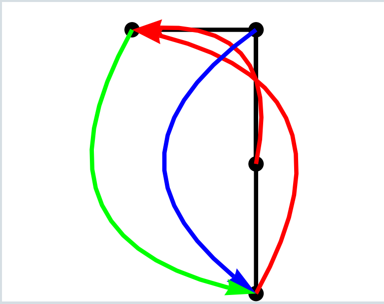
Four-person intermezzo



Four-person intermezzo, continued

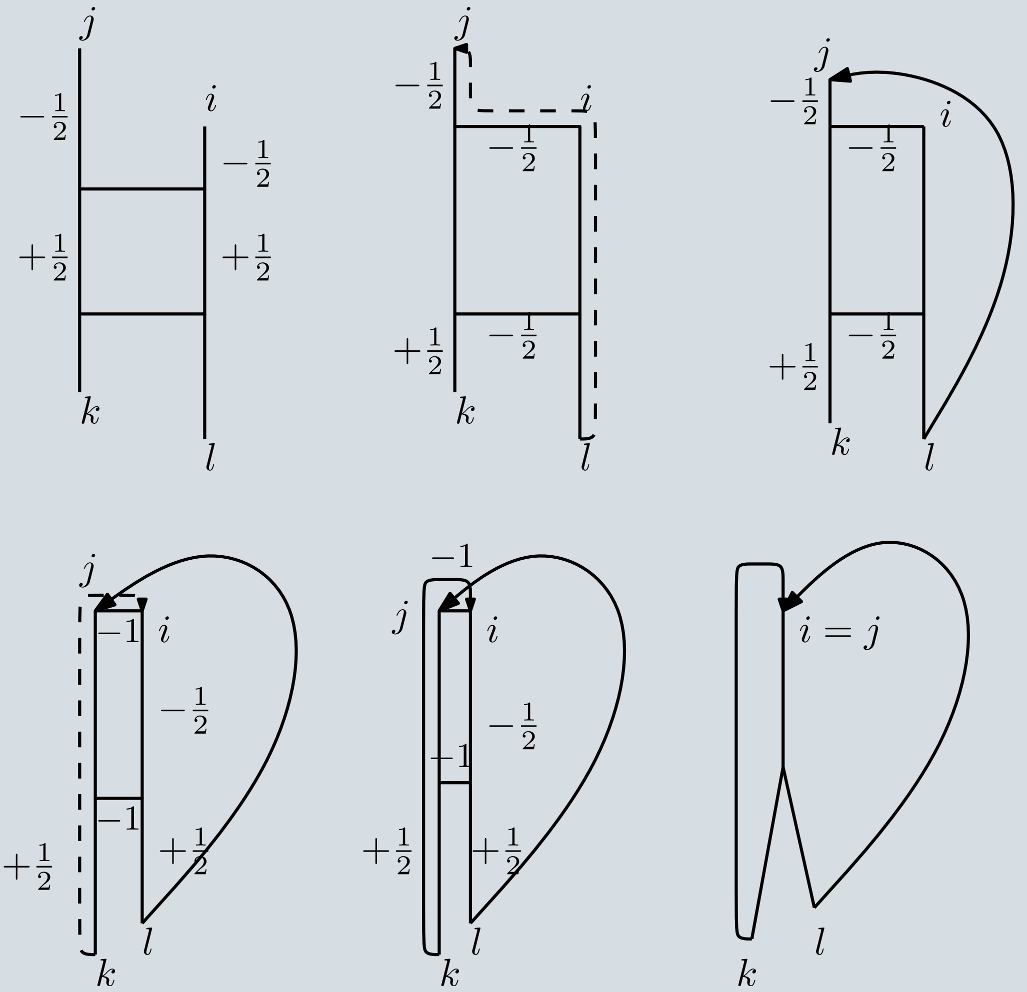


Four-person intermezzo, continued

-16 orbits of 6-dimensional cones (?)
-maximal length 6 (?)
-pure (?)

Four-person intermezzo, one parameter



Lossy gossip problems

-Combinatorial proof of
$\dim M_n = \binom{n}{2}$?
-Description of cones?
-Tight spans?
-Is M_n pure?
-Maximal length of an element?
-Efficient membership test?

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Proof of structure theorem

M_n is the support of a finite polyhedral fan of dimension $\binom{n}{2}$.

$$C_{24}(a) = \begin{bmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & a \\ \infty & \infty & 0 & \infty \\ \infty & a & \infty & 0 \end{bmatrix}$$
$$= \operatorname{Trop} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t^a) & 0 & -\sin(t^a) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(t^a) & 0 & \cos(t^a) \end{bmatrix}$$
$$\in O_4(\mathbb{C}\{\{t\}\})$$

$$\rightsquigarrow M_n \subseteq \operatorname{Trop}(O_n)$$

$\dim O_n = \binom{n}{2}$
apply Bieri-Groves $\square$