Finiteness results for infinite-dimensional varieties with many symmetries

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Fundamental theorem

Increasing maps

$\mathrm{Inc}(\mathbb{N}) = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(1) < \pi(2) < \ldots\}$

$\mathbb{C}[x_1, x_2, \ldots]$

$\pi x_i = x_{\pi(i)}$

Cohen (1967), Aschenbrenner-Hillar (2007)

$I_1 \subseteq I_2 \subseteq \ldots$ ideals in $\mathbb{C}[x_1, x_2, \ldots]$

$\mathrm{Inc}(\mathbb{N})$-stable

$\Rightarrow$ chain stabilises: $I_n = I_{n+1} = \ldots$

"$\mathrm{Inc}(\mathbb{N})$-Noetherian"

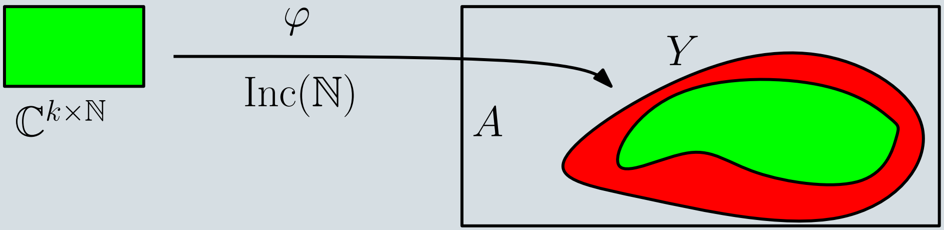
Cohen (1987), Hilllar-Sullivant (2009)

$\mathbb{C}\begin{bmatrix} x_{11} & x_{12} & \cdots \\ \vdots & \vdots & \\ x_{k1} & x_{k2} & \cdots \end{bmatrix}$ is $\mathrm{Inc}(\mathbb{N})$-Noetherian.

Applications

- Independent set theorem (Hillar-Sullivant 2009)
- **Relations among Vandermonde determinants** (Dress 1990s, D 2008)
- Gaussian k-factor model (Drton-Sturmfels-Sullivant 2007, D 2008)
- **Tensors of bounded rank** (Kuttler-D 2010)

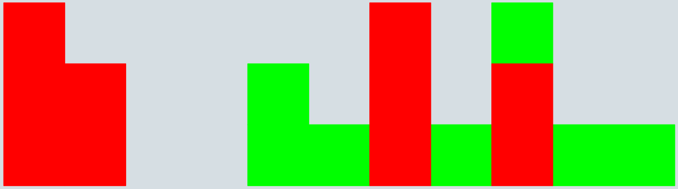
Bottom line



(φ, A) “reasonable” $\Rightarrow \overline{\mathrm{im}(\varphi)}$ defined by finitely many $\mathrm{Inc}(\mathbb{N})$-orbits of equations.

Proof of fundamental theorem

$\mathrm{Inc}(\mathbb{N})$**-divisibility**
u, v monomials in $x_1, x_2, \ldots$
$u|_{\mathrm{Inc}(\mathbb{N})}v :\Leftrightarrow \exists \pi \in \mathrm{Inc}(\mathbb{N}) : \pi u | v$



$$x_1^3\, x_2^2 \,|_{\mathrm{Inc}(\mathbb{N})}\, x_1^2\, x_2^1\, x_3^3\, x_4^1\, x_5^3\, x_6^1\, x_7^1$$

Monomial order
$\leq$ well-order
$u \leq v \Rightarrow su \leq sv$
$u \leq v \Rightarrow \pi u \leq \pi v$

Exercise
$u \leq \pi u$

Proof of fundamental theorem, cont.

Lemma
monomials $u_1, u_2, u_3, \ldots \Rightarrow \exists i < j : u_i|_{\mathrm{Inc}(\mathbb{N})} u_j$
("well-quasi-order")

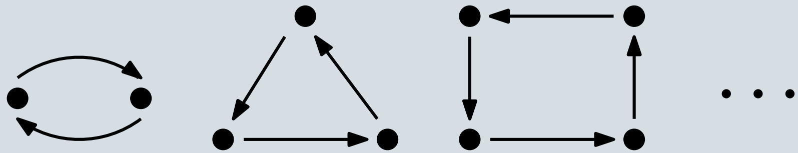
Proof
$u_1, u_2, \ldots$ $\leq$-minimal counter-example ("bad sequence")
$i_1 < i_2 < \ldots$ exponent of x_1 increasing weakly
$\Rightarrow u_1, \ldots, u_{i_1-1}, u'_{i_1}, u'_{i_2}, \ldots$ smaller counter-example
where $u_{i_j} = x_1^{e_{i_j}} \cdot (a \mapsto a+1) u'_{i_j}$ $\square$

Note
- Lemma $\Rightarrow$ fundamental theorem.
- Finite $\mathrm{Inc}(\mathbb{N})$-Gröbner basis!

Non-example

$\mathbb{C}[y_{ij} \mid i, j \in \mathbb{N}]$
$\pi y_{ij} = y_{\pi(i)\pi(j)}$
not $\mathrm{Inc}(\mathbb{N})$-Noetherian.

$I = \langle y_{12}y_{21}, y_{12}y_{23}y_{31}, y_{12}y_{23}y_{34}y_{41}, \ldots \rangle_{\mathrm{Inc}(\mathbb{N})}$
not finitely generated.



Relations among Vandermonde

$k, n \in \mathbb{N}$
$v_S := \prod_{i,j \in S, i>j} (x_i - x_j) \in \mathbb{C}[x_1, \ldots, x_n],\ S \in \binom{[n]}{k}$

Dress (1990s)
Polynomial relations among v_S stabilise for $n \to \infty$?

D (2009)
Yes!

Recall
$S = \{i_1 < \ldots < i_k\}$

$$v_S = \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_{i_1} & x_{i_2} & \ldots & x_{i_k} \\ x_{i_1}^2 & x_{i_2}^2 & \ldots & x_{i_k}^2 \\ \vdots & \vdots & & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \ldots & x_{i_k}^{k-1} \end{bmatrix}$$

Relations among Vandermonde, cont.

Proof
Pass to $n = \infty$.
($\varphi : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\binom{\mathbb{N}}{k}} =: A$ with coordinates y_S)

$\mathbb{C}[v_S | S \in \binom{\mathbb{N}}{k}]$ is quotient of
$R = \mathbb{C}[y_S | S \in \binom{\mathbb{N}}{k}] / \text{Plücker relations}$
(finitely many up to $\mathrm{Inc}(\mathbb{N})$; define Y).

$R = \mathbb{C}[z_{ij} | i \in [k],\ j \in \mathbb{N}]^{\mathrm{SU}_k}$
Reynolds: $\rho : \mathbb{C}[z_{ij} | i, j] \mapsto R,\ f \mapsto \int_{\mathrm{SU}_k} g f \mathrm{d}g$

$I_1 \subseteq I_2 \subseteq \ldots$ $\mathrm{Inc}(\mathbb{N})$-stable in R
$J_i = \mathbb{C}[z_{ij} | i, j] \cdot I_i$
fundamental theorem: $J_n = J_{n+1} = \ldots$
$\Rightarrow I_n = \rho(J_n) = \rho(J_{n+1}) = I_{n+1} = \ldots$ $\square$

Tensors of bounded rank

$V_1, \ldots, V_p$ vector spaces
$V_1 \otimes \cdots \otimes V_p$ tensor product

Rank ≤ 1
$\mu = v_1 \otimes \cdots \otimes v_p$

Rank $\leq k$
$\omega = \mu_1 + \ldots + \mu_k,\ \mu_i$ rank ≤ 1

Border rank $\leq k$
Zariski closure

Theorem (D-Kuttler, 2010)
$\forall k\, \exists d\, \forall p\, \forall V_1, \ldots, V_p$:
$\{$tensors of border rank $\leq k\}$
defined in degree $\leq d$.

Why border rank?

Example
$\omega \in \mathbb{C}^{2\times 2\times 2}$
$\psi : \mathbb{C}^2 \to \mathbb{C}^{2\times 2}$,

$$x \mapsto \begin{bmatrix} x_1\omega(1,1,1) + x_2\omega(1,1,2) & x_1\omega(1,2,1) + x_2\omega(1,2,2) \\ x_1\omega(2,1,1) + x_2\omega(2,1,2) & x_1\omega(2,2,1) + x_2\omega(2,2,2) \end{bmatrix}.$$

Cases
1. $\dim \operatorname{im} \psi = 0, 1$: $\operatorname{rk} \omega \leq 2$
2. $\operatorname{im} \psi = \langle A, B \rangle$ with $\operatorname{rk}(A) = \operatorname{rk}(B) = 1$: $\operatorname{rk} \omega = 2$
3. otherwise: ω has rank 3

Case 3: ω zero of discriminant$(\det \psi(x_1, x_2))$
Cayley's hyperdeterminant

$\rightsquigarrow \{\omega \mid \operatorname{rk} \omega \leq 2\}$ is complicated;
$\{\omega \mid \text{ border rank } \leq 2\} = \mathbb{C}^{2\times 2\times 2}$

Border rank ≤ 2

Flattening
$\flat : V_1 \otimes \cdots \otimes V_p \to (V_1 \otimes \cdots \otimes V_q) \otimes (V_{q+1} \otimes \cdots \otimes V_p)$
does not increase rank.

Theorem (Landsberg-Manivel 2004)
3×3-minors of flattenings
define "border rank ≤ 2".

Conjecture (Garcia-Stilmann-Sturmfels 2005)
Actually, generate their ideal.

Tensors of bounded rank, proof

May take all V_i equal to V of fixed dimension $n \geq k$
(Allman-Rhodes, Landsberg-Manivel-Weyman).

Pass to $p = \infty \rightsquigarrow A = \lim_{\leftarrow p} V^{\otimes p}$
$X^{\leq k} = \{\text{border rank} \leq k\} \subseteq A$.
$\Pi = \mathrm{Inc}(\mathbb{N}) \ltimes (\bigcup_n \mathrm{GL}(V)^n)$ natural symmetries

$Y^{\leq k}$ defined by $(k+1) \times (k+1)$-minors
(finitely many Π-orbits suffice)
$X^{\leq k} \subseteq Y^{\leq k}$

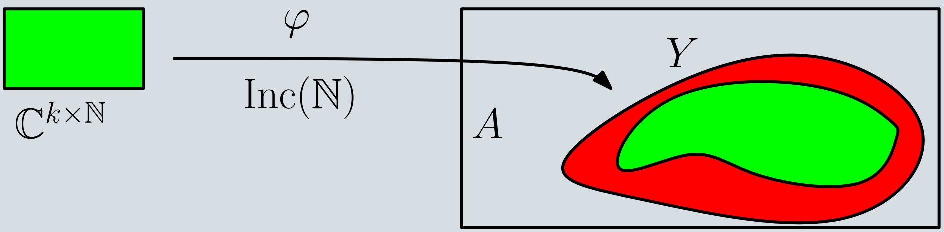
Claim:
$Y^{\leq k}$ is Π-Noetherian topological space
induction on k, uses fundamental theorem in each step.

$\rightsquigarrow X^{\leq k} \subseteq Y^{\leq k}$ defined by finitely many Π-orbits of equations. $\square$

Further issues

1. Buchberger algorithm
   (Cohen 1987, La Scala-Levandovskyy 2010, Brouwer-D 2010)
2. Application to GSS conjecture?
3. Other monoids and well-quasi-orders?
   (Higman!, Kruskal?, Robertson-Seymour??)
4. Over real numbers, Euclidean closure?
5. What is “reasonable” in the

Bottom line?



(A, φ) “reasonable” $\Rightarrow$ image defined by finitely many $\mathrm{Inc}(\mathbb{N})$-orbits of equations.