Symmetries and ∞-dim limits of algebro-statistical models

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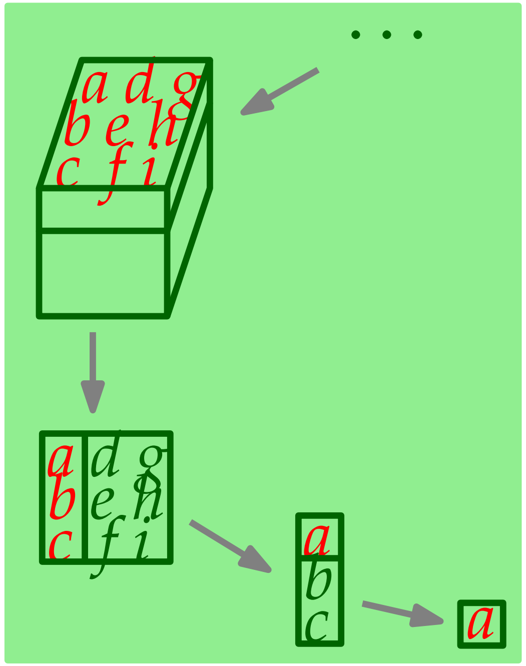

Jan Draisma

TU Eindhoven

Part I: some infinite-dimensional commutative algebra

What is an infinite-dimensional variety?

inductive limits of finite-dimensional varieties,
projective limits, spectra of infinite-dimensional rings, etc.

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V: countable-dimensional space over $\mathbb{C}$ (or $\mathbb{R}$) of *coordinates*
V^*: dual space, topological space with Zariski topology
Closed subsets $X \subseteq V^*$ are called *infinite-dimensional varieties*.

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Sequence model

If $V_1 \subseteq V_2 \subseteq \ldots$ finite-dimensional with $V = \bigcup_i V_i$, then
$V^* = \lim_{\leftarrow} V_i^*$ with $V_1^* \twoheadleftarrow V_2^* \twoheadleftarrow \cdots$
(*both as set and as topological space*)

Noetherianity modulo group actions

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$\rightsquigarrow$ G acts on SV by algebra auto and on V^* by homeo.

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(Non-)Examples of G-Noetherianity

Finite-by-infinite matrices

Fix $k \in \mathbb{N}$;

$\mathrm{Sym}(\mathbb{N})$ acts on $V = \langle x_{ij} \mid i \in [k], j \in \mathbb{N} \rangle$

by $\pi(x_{ij}) = x_{i\pi(j)}$.

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$\mathrm{Sym}(\mathbb{N})$ acts by $\pi(x_{ij}) = x_{\pi(i),\pi(j)}$
$\rightsquigarrow \mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$ is *not* $\mathrm{Sym}(\mathbb{N})$-Noetherian;
e.g. the $\mathrm{Sym}(\mathbb{N})$-stable ideal generated by
$x_{12}x_{21},\ x_{12}x_{23}x_{31},\ x_{12}x_{23}x_{34}x_{41},\ \ldots$
is not $\mathrm{Sym}(\mathbb{N})$-finitely generated.

(*neither* $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$-*Noetherian*)

(Non-)Examples, continued

Theorem (*Matrices of bounded rank*)
$\mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]/(\text{all } (k+1) \times (k+1)\text{-subdeterminants})$
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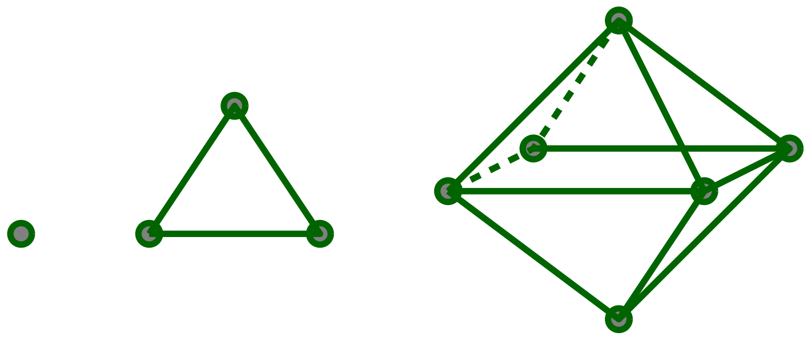
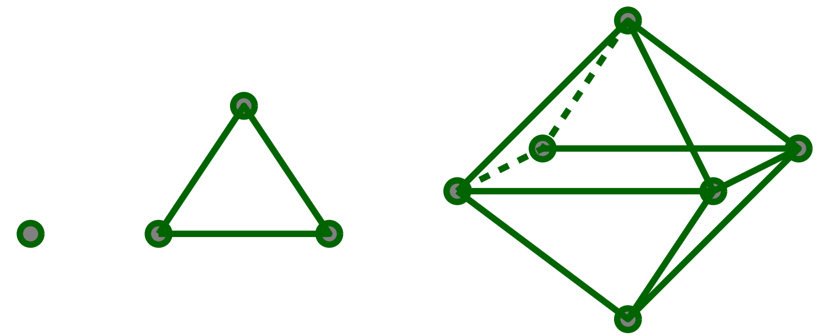
Theorem [D-Eggermont 2014]
$(\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^p$ is $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian for each p, via
$(g, h) \circ (x, \ldots, z) := (gxh^{-1}, \ldots, gzh^{-1})$.

Example: second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



Finitely generated symmetric Markov bases



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$v_{ij} = v_{ji}$ and $v_{ij} + v_{kl} = v_{il} + v_{kj}$ for i, j, k, l distinct

$\rightsquigarrow$ if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$,
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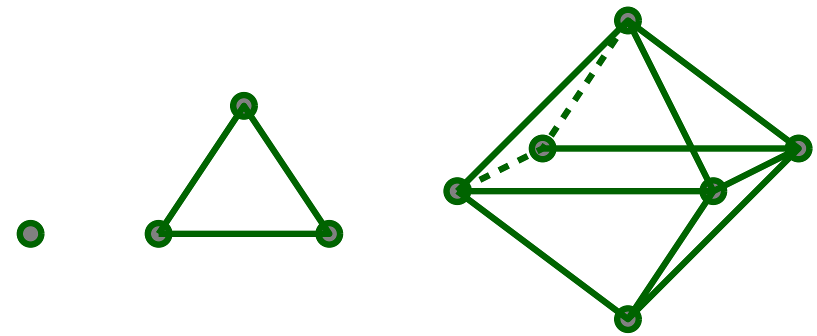

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For *any* family $(P_n \subseteq \mathbb{Z}^F \times \mathbb{Z}^{k \times n})$, F finite, if $P_n = \mathrm{Sym}(n) P_{n_0}$ for $n \geq n_0$, then $\exists n_1$: for $n \geq n_1$ has a Markov basis M_n with $M_n = \mathrm{Sym}(n) M_{n_0}$.

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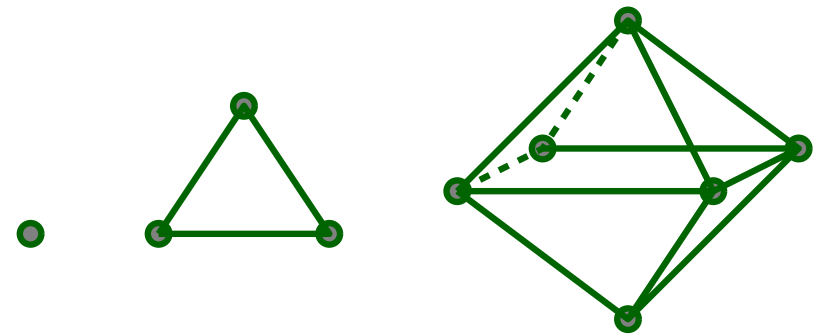

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Explicit results for width $n_0 = 2$: [Kahle-Krone-Leykin 2014]

Part II: Applications to algebro-statistical models

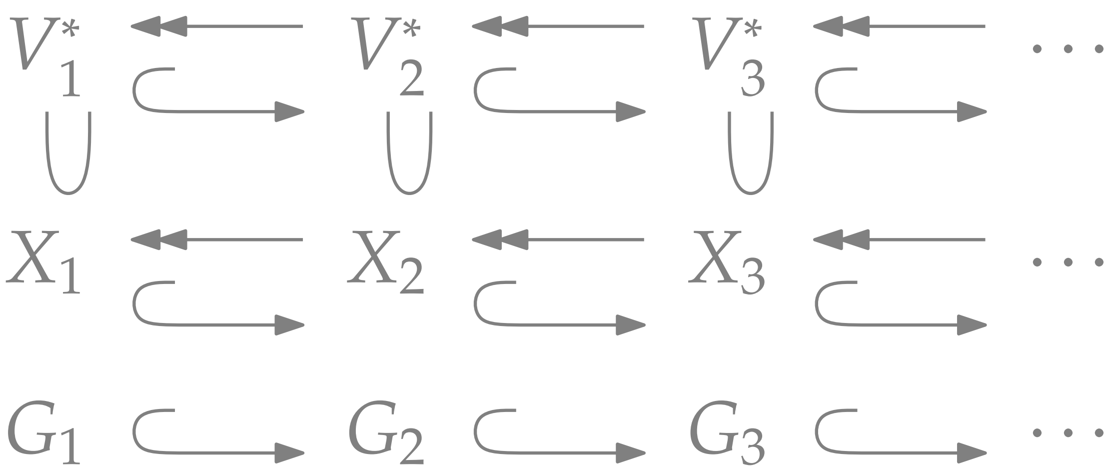
Setting

$V_1^*, V_2^*, \ldots$ fin-dim spaces; $X_i \subseteq V_i^*$ subvariety

G_i group acting linearly on V_i^* preserving X_i

$G_i \subseteq G_{i+1}$ & maps $\pi : V_{i+1}^* \to V_i^*$ and $\iota : V_i^* \to V_{i+1}^*$ both

G_i-equivariant, mapping X_{i+1} into X_i and v.v. & $\pi \circ \iota = \mathrm{id}$



Constructing limits

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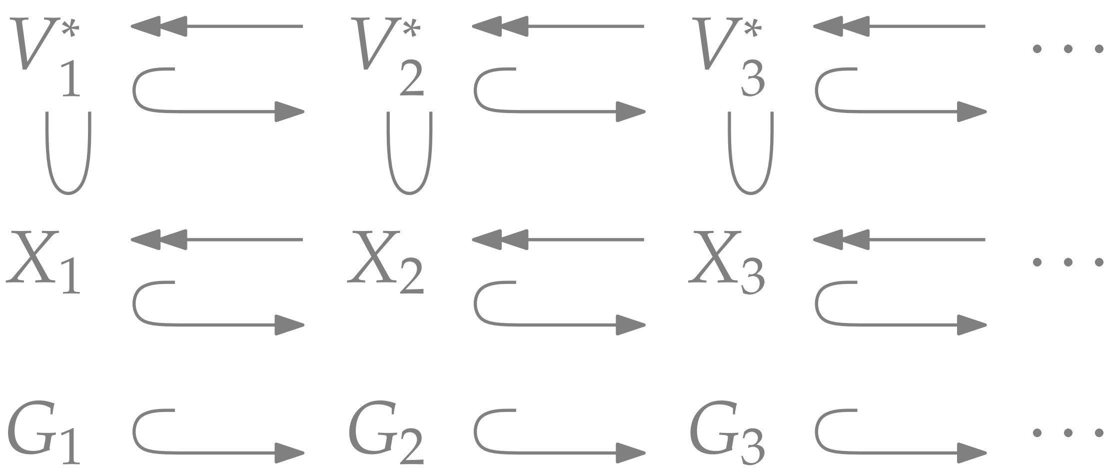

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Sequence $(X_i \subseteq V_i^*)_i$ *stabilises* if for $n \gg 0$:
$p \in V_n^*$ lies in X_n
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$$\begin{array}{ccccccc}
V_1^* & \twoheadleftarrow \atop \hookrightarrow & V_2^* & \twoheadleftarrow \atop \hookrightarrow & V_3^* & \twoheadleftarrow \atop \hookrightarrow & \cdots \\
\cup & & \cup & & \cup & & \\
X_1 & \twoheadleftarrow \atop \hookrightarrow & X_2 & \twoheadleftarrow \atop \hookrightarrow & X_3 & \twoheadleftarrow \atop \hookrightarrow & \cdots \\
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G_1 & \hookrightarrow & G_2 & \hookrightarrow & G_3 & \hookrightarrow & \cdots
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$\rightsquigarrow V_\infty^* := \lim_\leftarrow V_n$; $X_\infty := \lim_\leftarrow X_n$; $G_\infty := \cup_n G_n$

Lemma Stabilisation is "equivalent" to: $X_\infty \subseteq V_\infty^*$ is defined by finitely many G_∞-orbits of equations.

Fixed row and column sums

$A, B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with $a_{i+} = b_{i+}$ and $a_{+j} = b_{+j}$

$\Rightarrow \exists A = A_0, A_1, \ldots, A_k = B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with

$A_l - A_{l-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightsquigarrow$ moves "independent" of m, n.

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Theorem [Diaconis-Sturmfels 1998]

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Conjecture [Hoşten-Sullivant 2007]

Similar stabilisation conjecture for Markov basis for sampling higher-dimensional contingency tables.

The independent set theorem, continued

Hierarchichal models

F family of subsets of $[m]$

$y(i_1, \ldots, i_m)$ and $x(S, (i_s)_{s \in S})$ for $S \in F$ variables

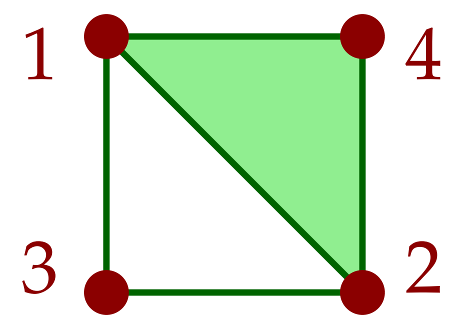
$I := \ker[y(i_1, \ldots, i_m) \mapsto \prod_{A \in S} x(S, (i_s)_{s \in S})]$

Example

$m = 4, F = \{124, 13, 23\}$

variables $y(abcd), x(abd), z(ac), u(bc)$

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$



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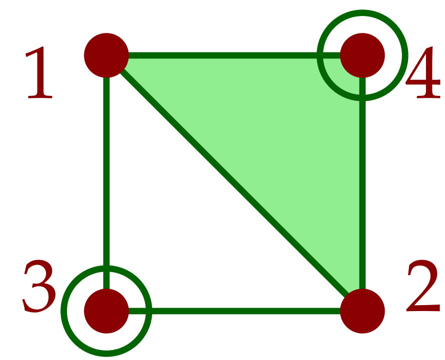

Theorem [Hillar-Sullivant 2012]

If $T \subseteq [m]$ *independent set* ($|T \cap S| \leq 1$ for $S \in F$);

$i_t, t \in T$ run through $\mathbb{N}$ and $i_t, t \notin T$ through $[r_t]$

$\rightsquigarrow I$ generated by finitely many $\mathrm{Inc}(\mathbb{N})$-orbits

(now this also follows from D-Eggermont-Krone-Leykin)

II: Cloning sinks in a Gaussian Bayesian model

G: directed acyclic graph on [n]
$X_i, i \in [n]$: jointly Gaussian
$X_j = \sum_{i \in \mathrm{pa}(j)} \lambda_{ij} X_i + a_j \epsilon_j$ where the $\epsilon_j \sim N(0,1)$ independent
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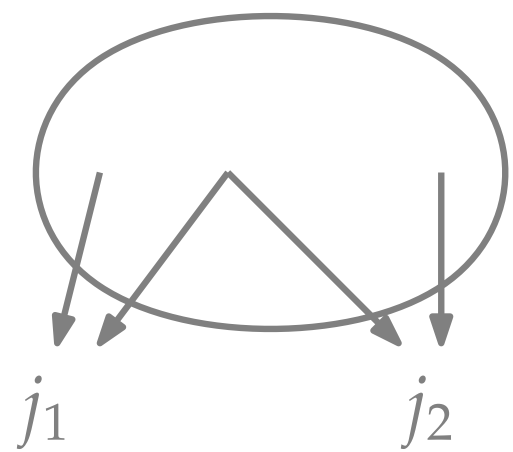

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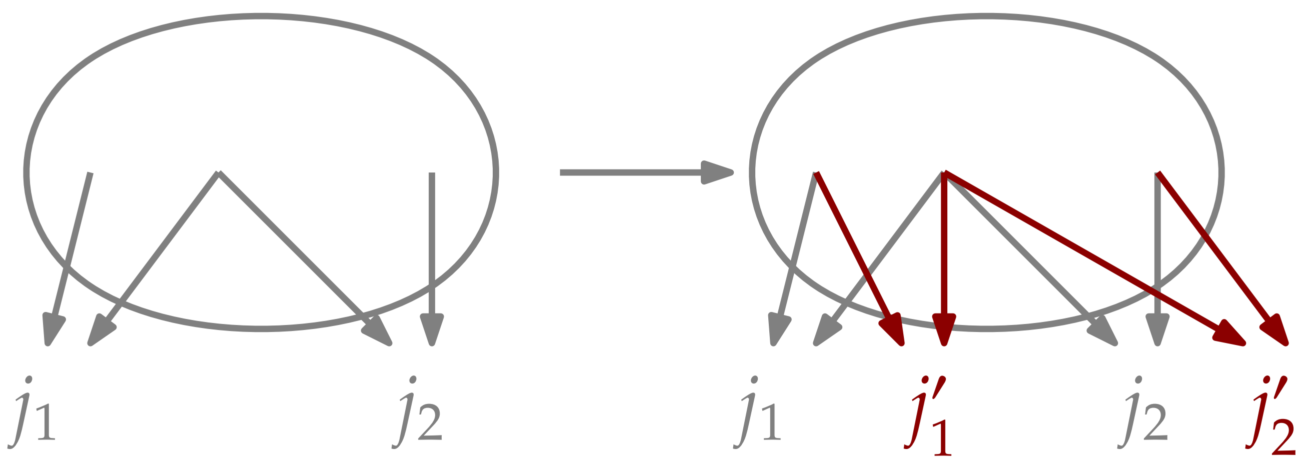

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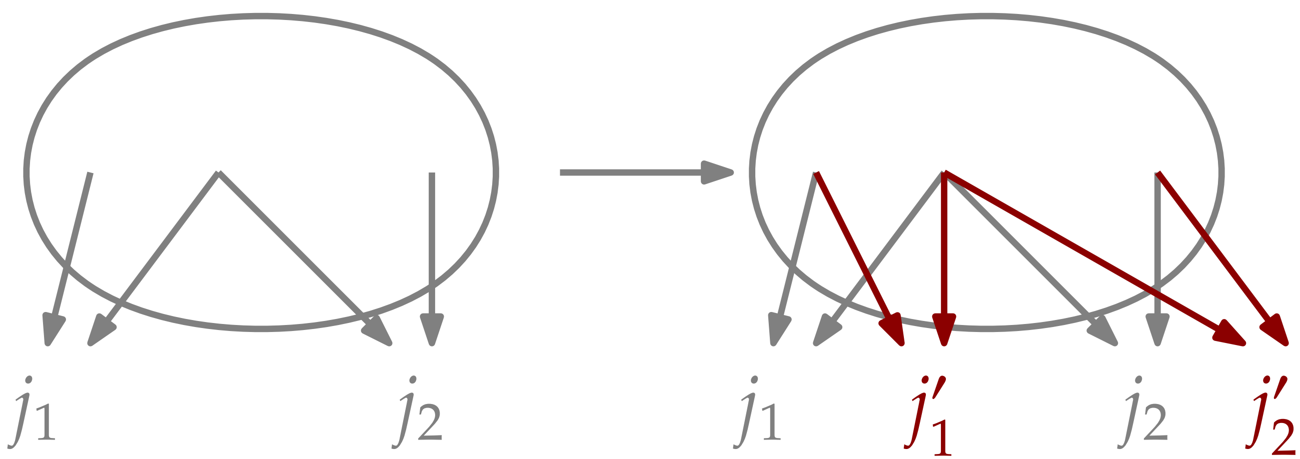

Theorem ["D 2010"]
Model stabilises under cloning sinks (via permuting clones).

Stabilisation for parameterised graphical models

undirected	DAG with hidden vars	
$M(G) := \{(p_{i_1,\ldots,i_n} = \prod_C \theta^C_{\mathbf{i}_C})_{i_1,\ldots,i_n}\} \subseteq \mathbb{C}^R$ prod over all cliques C $\theta^C \in \mathbb{C}^{R_C}$ *I: stabilises under increasing r_j for j in independent set*		*discrete* $R = \prod_j [r_j]$
		Gaussian mean 0

Stabilisation for parameterised graphical models

undirected	DAG with hidden vars	
$M(G) := \{(p_{i_1,\ldots,i_n} = \prod_C \theta^C_{\mathbf{i}_C})_{i_1,\ldots,i_n}\} \subseteq \mathbb{C}^R$ prod over all cliques C $\theta^C \in \mathbb{C}^{R_C}$ *I: stabilises under increasing r_j for j in independent set*		*discrete* $R = \prod_j [r_j]$
	$\{\Sigma = (I-\Lambda)^{-T} D (I-\Lambda)^{-1}\}$ $\Lambda_{ij} = 0$ if $i \not\to j$ $M(G) \subseteq \mathbb{C}^{([n]-H)\times([n]-H)}$ *II: stabilises under cloning sinks*	*Gaussian* *mean* 0

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$M(G) := \{\Sigma = K^{-1}\}$ $K_{ij} = 0$ if $ij \notin E(G)$ $M(G) \subseteq \mathbb{C}^{n\times n}$ *stabilisation??*	$\{\Sigma = (I-\Lambda)^{-T} D (I-\Lambda)^{-1}\}$ $\Lambda_{ij} = 0$ if $i \not\to j$ $M(G) \subseteq \mathbb{C}^{([n]-H)\times([n]-H)}$ *II: stabilises under cloning sinks*	*Gaussian mean 0*

Definition

$C \in \mathbb{R}_{\geq 0}^{m \times n} \rightsquigarrow \mathrm{rk}_{\geq 0} C := \min\{r \mid \exists (A, B) \in \mathbb{R}_{\geq 0}^{m \times r} \times \mathbb{R}_{\geq 0}^{r \times n} : C = AB\}$

IV: Nonnegative matrix rank

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Observation/Theorem [Kubjas, Robeva, Sturmfels 2013]

EM-algorithm for $M_r^{m\times n}$ often converges to boundary!

Explicit, quantifier-free expression for $r = 2$.

Algebraic boundary

$\overline{\partial M_r^{m\times n}}$: Zariski closure $\subseteq \mathbb{C}^{m\times n}$
hypersurface in the variety of rank-r matrices

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Theorem [Kubjas-Robeva-Sturmfels 2013]

Apart from coordinate hyperplanes, for $m, n \geq 4$, $\overline{\partial M_3^{m\times n}}$ has 2 $\mathrm{Sym}(m) \times \mathrm{Sym}(n)$-orbits of irreducible components, parameterised by the following and its transpose:

$$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \\ * & * & * \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & * & * & * \\ * & * & 0 & * & * \\ * & * & * & 0 & * \end{bmatrix}$$

Nonnegative matrix rank, continued

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This component has a GB of 4×4 minors plus $\binom{m}{3}$ sextics.

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Now a **Theorem** *due to Eggermont-Horobeţ-Kubjas.*
But what about higher nonnegative rank??

Conclusions and questions

- Many algebro-statistical models fit into families with a meaningful limit.
- There is an ever growing body of commutative algebra for dealing with these limits up to symmetry.
- Do discrete Bayesian models stabilise under cloning sinks?
- Do undirected Gaussian graphical models exhibit any kind of stabilisation?
- If you have other families of models where you expect stabilisation, come talk to me!

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Děkuji!