Classical theory of Lie algebras

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Lie algebra

vector space L with
bilinear *bracket*

$$[\,.\,,\,.\,] : L \times L \to L$$

subject to *anti-commutativity*

$$[x, x] = 0 \qquad (\leadsto [x, y] = -[y, x])$$

and *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

First construction: linear maps

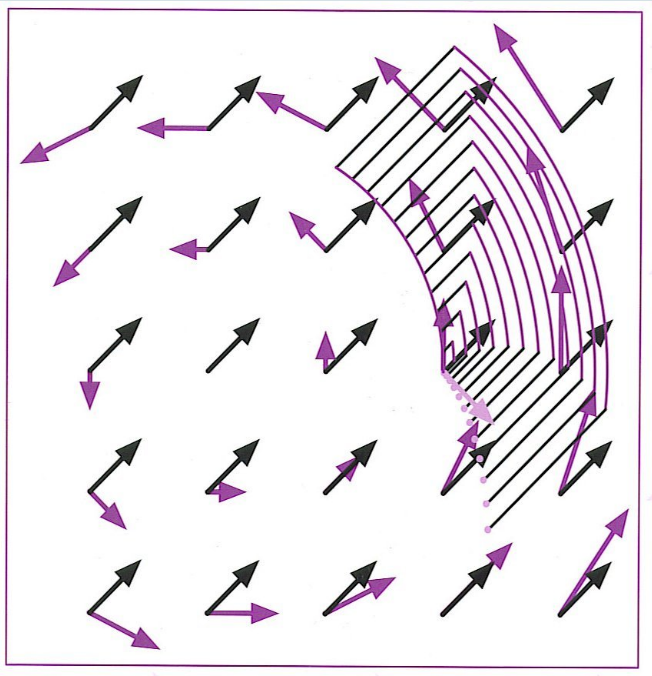
$L := \{\text{linear maps on a vector space } V\} =: \mathrm{End}(V)$
$[A, B] := AB - BA$ *commutator*
anti-commutativity

$$[A, A] = AA - AA = 0$$

Jacobi identity

$$\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
=&ABC - ACB - BCA + CBA \\
+&BCA - CBA - CAB + BAC \\
+&CAB - BAC - ABC + ACB \\
=&0
\end{aligned}$$

Second construction: vector fields



X, Y vector fields on $\mathbb{R}^n$
$\rightsquigarrow$ so is $[X, Y]$

operation (derivation) on functions

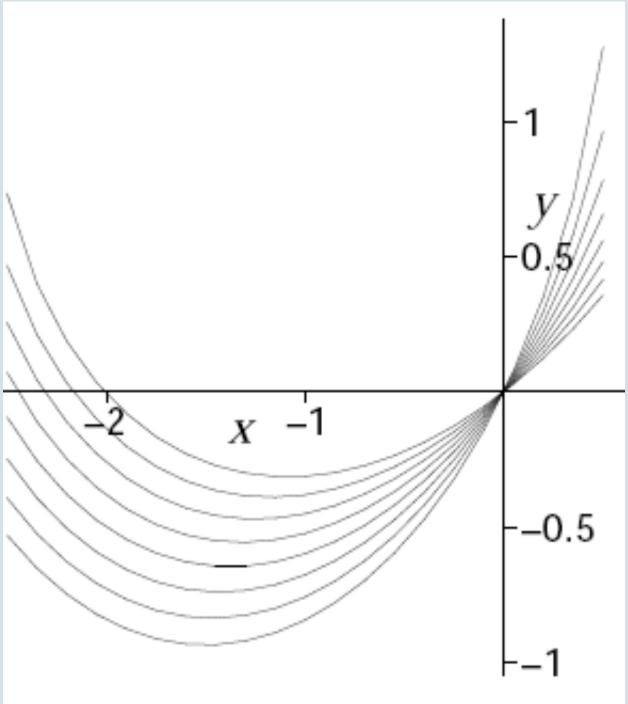
$$X(f)(p) = (df)(X(p))$$

bracket $\leftrightsquigarrow$ commutator

$$[x\partial_y - y\partial_x, \partial_x + \partial_y] = \partial_x - \partial_y$$

Jacobi identity follows!

Vector fields as infinitesimal symmetries



ODE

$$y^{(4)} - \frac{5(y^{(3)})^2}{3y^{(2)}} - (y^{(2)})^{5/3} = 0$$

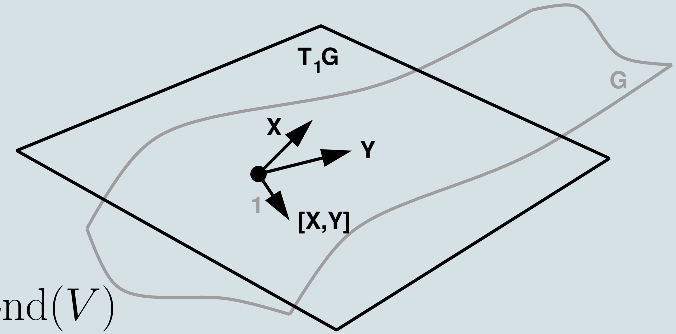
infinitesimal rotational symmetry

$$x\partial_y - y\partial_x$$

Sophus Lie ($\sim$ 1890): ODE
$\rightsquigarrow$ algebra of vector fields in $\mathbb{R}^2$
classified up to diffeomorphisms
analogy with Galois theory?

$\rightsquigarrow$ helps solving (computer algebra)

Third construction: Lie algebras from groups



V finite-dimensional vector space
$\mathrm{GL}(V) :=$ {invertible $g \in \mathrm{End}(V)$}
G closed subgroup of $\mathrm{GL}(V)$

$T_I G =$ tangent space at $I \in G \subseteq \mathrm{End}(V)$
fact: closed under commutator
$\rightsquigarrow \mathfrak{g} := (T_I(G), [\,.\,,\,.\,])$ *Lie algebra of* G

properties of $G \rightsquigarrow$ properties of $\mathfrak{g}$
e.g. G Abelian $\Rightarrow [\mathfrak{g}, \mathfrak{g}] = 0$

Lie algebras from classical groups

B/D
non-degenerate symmetric bilinear form on V
$G := \{g \mid g^T g = I\} =: \mathrm{O}(V)$
$\rightsquigarrow \mathfrak{g} = \{X \mid (I + \epsilon X)^T (I + \epsilon X) = I \mod \epsilon^2\} = \{X \mid X^T + X = 0\}$
$\rightsquigarrow \mathfrak{o}(V) = \mathfrak{so}(V)$
note $(XY - YX)^T = Y^T X^T - X^T Y^T = -(XY - YX)$

A
$\mathrm{SL}(V) := \{\text{determinant } 1\} \rightsquigarrow \mathfrak{sl}(V) := \{\text{trace } 0\}$

C
$\mathrm{Sp}(V)$, $\mathfrak{sp}(V)$ like $\mathrm{O}(V)$, $\mathfrak{o}(V)$ but skew form

Intermezzo: exponential map

Lie algebra elements are "infinitesimal group elements"
often $x \in L \rightsquigarrow \exp(tx) = 1 + tx + \frac{t^2}{2!}x^2 + \ldots$ "real group element"
1-parameter group: $\exp((s+t)x) = \exp(sx)\exp(tx)$
characteristic zero needed?

examples
$L = \mathfrak{g} \subseteq \mathfrak{gl}(\mathbb{C}^n) \rightsquigarrow \exp(tx) = e^{tx}$
L of vector fields $\rightsquigarrow \exp(tx)$ is the flow

open problem
$V = p(x,y,z)\partial_x + q(x,y,z)\partial_y + r(x,y,z)\partial_z$
p, q, r polynomials $\in \mathbb{Q}[x,y,z]$
decide algorithmically:

$$\exp(tV)x = x + tp + \frac{t^2}{2!}Vp + \ldots \text{ polynomial in } t?$$

Theory

goal (this talk): classification
finite-dimensional simple Lie algebras
certain Lie algebras of vector fields

tool: representation theory
$L \longrightarrow \mathrm{End}(V)$?
$L \longrightarrow \{\text{vector fields}\}$?

tool: structure theory
ideals
subalgebras
simple Lie algebras

Structure theory: ideals and simple Lie algebras

$I \subseteq L$ *ideal* if $[I, L] \subseteq I$
($\rightsquigarrow$ normal subgroups)

L *simple* if 0 and L are the only ideals
(and L not 1-dimensional)

$\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, $\mathfrak{sp}(V)$ are (usually) simple

other finite-dimensional ones?

Representation theory: adjoint representation

Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$

$\Leftrightarrow$

$$[[y,z],x]=[y,[z,x]]-[z,[y,x]]$$

$\Leftrightarrow \mathrm{ad}: L \longrightarrow \mathrm{End}(L),\ y \mapsto [y,\ .\ ]$ intertwines bracket and commutator

$$\mathrm{ad}([y,z])=[\mathrm{ad}(y),\mathrm{ad}(z)]$$

ad is the *adjoint representation* of L
(with kernel $\{x \in L \mid [x,L]=0\}$, the *center*)

fundamental tool in structure theory and classification!

Structure theory: Cartan subalgebras

L simple finite-dimensional over $\mathbb{C}$
Cartan subalgebra H: maximal with H Abelian and all $\mathrm{ad}_L(h)$ diagonalisable
$\rightsquigarrow ad_L(H)$ simultaneously diagonalisable

H unique up to (inner) automorphisms of L

$L = \mathfrak{sl}_3 = \mathfrak{sl}(\mathbb{C}^3)$

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \right\} \text{ ad-diagonalisable, e.g.}$$

$$\left[\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = (a-b) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

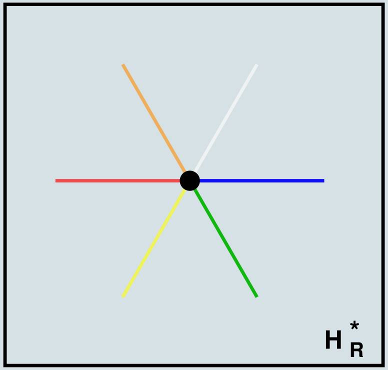
Structure theory: root systems

H Cartan subalgebra in L
$\rightsquigarrow L$ direct sum of common eigenspaces $L_\alpha,\ \alpha \in H^*$
$\Phi(L) := \{\text{all such } \alpha\} \setminus \{0\}$ *root system* ($L_0 = H$)

$L = \mathfrak{sl}_3 \rightsquigarrow |\Phi| = 6$:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

eigenspaces



root system Φ

fact: every *root space* $L_\alpha, \alpha \in \Phi$ 1-dimensional

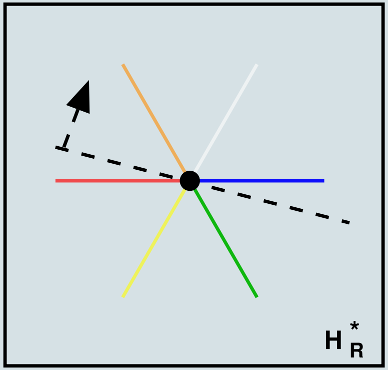
Classification

abstract root system:
finite set in Euclidean space, with certain axioms

Cartan (1894):
$L \mapsto \Phi(L)$ is bijection
$\{$ finite-dimensional complex simple Lie algebras $\} \to \{$ root systems $\}$
root systems classified

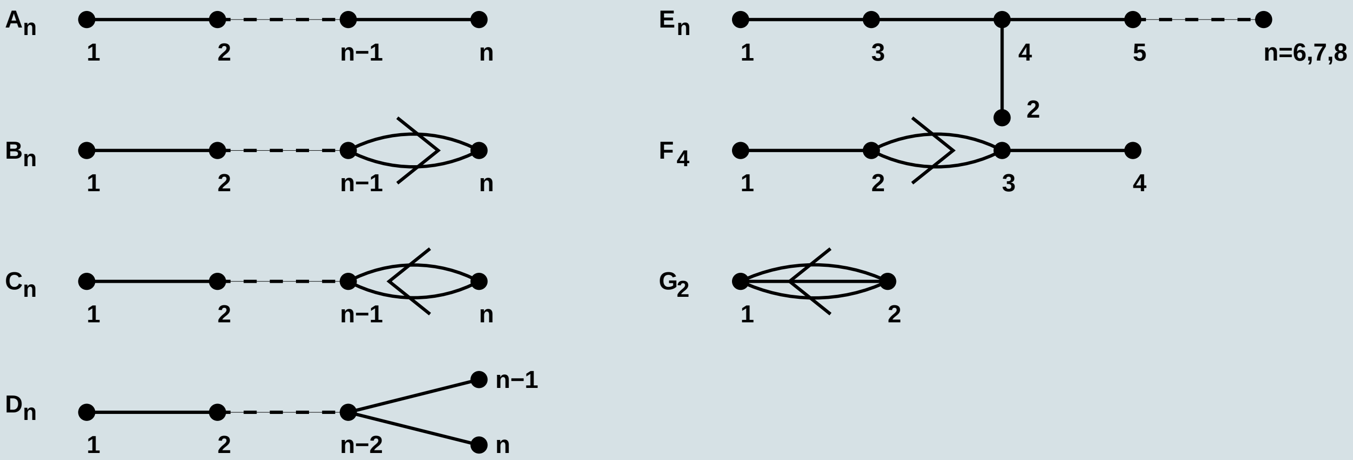
root systems classified combinatorially through *Dynkin diagrams*:

$L = \mathfrak{sl}_3 \rightsquigarrow$



$\rightsquigarrow$

Classification: root systems



all are Lie algebras of groups

$A_n \leftrightsquigarrow \mathfrak{sl}_{n+1}$

$B_n \leftrightsquigarrow \mathfrak{so}_{2n+1}$

$C_n \leftrightsquigarrow \mathfrak{sp}_{2n}$

$D_n \leftrightsquigarrow \mathfrak{so}_{2n}$

Representation theory: vector fields

$L \supseteq M$ Lie algebras
$m := \mathrm{codim}_L M < \infty$

Guillemin/Sternberg/Blattner (1960s):
$L \to \{$ (formal) vector fields in m variables $\}$
s.t. M stabiliser of 0

$$L = \mathfrak{sl}_3 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\}, M = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

$\rightsquigarrow \partial_1, \partial_2,$
$x_1\partial_2, 2x_1\partial_1 + x_2\partial_2, -x_1\partial_1 + x_2\partial_2, x_2\partial_1$
$-x_1^2\partial_1 - x_1x_2\partial_2, -x_1x_2\partial_1 - x_2^2\partial_2$

(L, M) *primitive* if $\nexists N : L \supsetneq N \supsetneq M$

Classification: infinite-dimensional primitive Lie algebras

Cartan (Guillemin,Sternberg,. . .) classified
infinite-dimensional primitive pseudo-groups:
L complex infinite-dimensional, $\operatorname{codim}_L M = m < \infty$
(L, M) primitive (+technical conditions)

$\rightsquigarrow$ six possibilities:
$L \cong W((m))$:= all formal vector fields
$L \cong (C)S((m))$:= those fixing a volume form (up to a constant)
$L \cong (C)H((2r))$:= those fixing a symplectic form (up to a constant)
$L \cong K((2r+1))$:= those leaving a contact structure

e.g. $S((m)) = \{X = \sum_{i=1}^{m} f_i \partial_i \mid \operatorname{div} X := \sum_i \partial_i(f_i) = 0\}$

Characteristic p and divided powers

$\operatorname{codim}_L M = m$
over $\mathbb{C}$:

$$L \longrightarrow \{\text{derivations of } \mathbb{C}[[x_1, \ldots, x_m]]\}$$

characteristic p:

$$L \longrightarrow \{\text{derivations of } \mathcal{O}((m))\}$$

$$\mathcal{O}((m)) := \{\sum_{r \in \mathbb{N}^m} c_r x^{(r)}\} \quad x^{(r)} x^{(s)} := \frac{(r+s)!}{r!s!} x^{(r+s)}$$

(think of $x^{(r)}$ as $\frac{x^r}{r!}$)
$\mathcal{O}((m))$ has nice finite-dimensional subalgebras
e.g. $r_i < p,\ s_i < p,\ r_i + s_i \geq p \Rightarrow x^{(r)} x^{(s)} = 0$

$\rightsquigarrow$ finite-dimensional versions of $W, (C)S, (C)H, K$ in char p
these are simple; are these + the classical ones all?

Wrapping up

take away:

1. finite-dimensional simple Lie algebras $/\mathbb{C}$ classified (combinatorially): 4 infinite series A_n, B_n, C_n, D_n, 5 exceptional Lie algebras E_6, E_7, E_8, F_4, G_2
2. infinite-dimensional primitive Lie algebras of vector fields $/\mathbb{C}$ classified; finite-dimensional analogues in char p
3. Lie algebra elements are "infinitesimal group elements"; exponential map "integrates" them

many subjects not touched upon: geometry, physics, Lie super-algebras, algorithms . . .