Noetherianity up to Sym and GL

(Joint with Chiu-Danelon-Eggermont-Farooq, . . .)

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Jan Draisma
University of Bern

Cohen's theorem

Setting. R a Noetherian ring, $k \geq 0$
Sym $= \bigcup_{n=0}^{\infty}$ **Sym**$(n) \curvearrowright R[x_{ij} \mid i \in [k], j \in \mathbb{N}]$: $\pi x_{ij} = x_{i\pi(j)}$

Theorem [Daniel Cohen]
Any chain $I_1 \subseteq I_2 \subseteq \cdots$ of **Sym**-stable ideals in $R[(x_{ij})_{i,j}]$ is eventually constant.

Generalisations and applications, e.g. in algebraic statistics:
[Hillar-Sullivant], [Nagel-Römer], [Alexandr-Kileel-Sturmfels], ...

Proof uses Higman's lemma: the partial order $x^{\alpha}|_{\mathrm{Inc}(\mathbb{N})} x^{\beta} \Leftrightarrow$ $\exists \pi \in \mathrm{Inc}(\mathbb{N}) : \pi x^{\alpha} | x^{\beta}$ does not have infinite antichains.

Dream
Use the algebraic geometry inspired by Cohen to prove non-existence of infinite antichains elsewhere in combinatorics.

GL-varieties

Setting [Snowden]

K an infinite field, $\mathbf{GL} = \bigcup_{n=0}^{\infty} \mathbf{GL}_n \curvearrowright \mathbf{V} = \langle x_1, x_2 \ldots \rangle_K$

A *polynomial* **GL***-representation* is a subquotient of a (possibly infinite) direct sum of tensor powers of $\mathbf{V}$.

A **GL***-algebra* has compatible structures of polynomial **GL**-representation and commutative algebra.

A(n affine) **GL**-*variety* is the spectrum of a *reduced* **GL**-algebra generated by finitely many **GL**-orbits.

Example

$A = S(S^2\mathbf{V}) = K[y_{ij} \mid i,j \in \mathbb{N}]/(y_{ij} - y_{ji})_{i,j} \rightsquigarrow$ the points of $\mathrm{Spec}(A)$ are symmetric $\mathbb{N} \times \mathbb{N}$-matrices.

Theorem [D]

For any **GL**-variety X, every chain $X_1 \supseteq X_2 \supseteq \ldots$ of **GL**-stable closed subvarieties is eventually constant.

Sample application to "slice rank"

On input an order-3 tensor $T \in K^n \otimes K^n \otimes K^n$, testing whether it admits a decomposition $T = u_1 \otimes A_{23} + A_{12} \otimes u_3$ over $\overline{K}$ can be done in polynomial time.

Important tool: shifting

Restricting the action to $\mathbf{GL} \to \mathbf{GL}, g \mapsto \begin{pmatrix} I_n & 0 \\ 0 & g \end{pmatrix}$ turns a **GL**-variety X into a new **GL**-variety $\mathrm{Sh}_n X$.

For symmetric matrices: $\mathrm{Sh}_n X \ni (M_{11}, (u_1, \ldots, u_n), M_{22})$ with M_{11} symmetric $n \times n$, $u_i \in K^{\mathbb{N}}$, $M_{22} \in X$.

Setting. Assume $\operatorname{char} K = 0$. Then each polynomial representation is a direct sum of $S^\lambda(\mathbf{V})$ (Schur functors). Write $\mathbf{A}^\lambda := \operatorname{Spec}(S(S^{\lambda_1}\mathbf{V} \oplus \cdots \oplus S^{\lambda_i}\mathbf{V}))$; e.g. $\mathbf{A}^{[(2)]}$ is the space of infinite symmetric matrices.

Shift theorem [Bik-D-Eggermont-Snowden]
For any **GL**-variety $X \neq \emptyset$ there exist $n \geq 0$ and nonzero $h \in K[X]$ such that $(\operatorname{Sh}_n X)[1/h] \cong B \times \mathbf{A}^\lambda$.

Example
Take $X \subseteq \mathbf{A}^{[(2)]}$ the rank $\leq n$-locus and $h :=$ the top left $n \times n$-subdeterminant. Then $(\operatorname{Sh}_n X)[1/h] \cong B \times \mathbf{A}^{[(1),\ldots,(1)]}$.

$\rightsquigarrow$ Rich structure theory: existence of curves, irred. X admits a surjective morphism $B \times \mathbf{A}^\lambda \to X$ with B irreducible, ...

“GCD” of the two theorems

Setting

R a K-algebra of finite type

$A := R \otimes S(\mathbf{V}^{\oplus k}) = R[x_{ij} \mid i \in [k], j \in \mathbb{N}]$

Corollary

Every chain $I_1 \subseteq I_2 \subseteq \ldots$ of **GL**-stable radical ideals in A is eventually constant.

Proof using Cohen’s theorem

The I_i are in particular **Sym**-stable (don’t need radical). □

Proof using GL-Noetherianity

A is a **GL**-algebra generated by finitely many **GL**-orbits; the I_i correspond to closed **GL**-subvarieties of $\mathrm{Spec}(A)_{\mathrm{red}}$. □

“LCM” of the two theorems

Setting

Assume char $K = 0$. X a **GL**-variety $\rightsquigarrow$ **Sym** $\times$ **GL** $\curvearrowright X^{\mathbb{N}}$

Simplest example: $\mathbb{N} \times \mathbb{N}$-matrices with **Sym** permuting rows and **GL** acting by linear transformations from the right.

Theorem [Chiu-Danelon-D-Eggermont-Farooq]

Let X be a **GL**-variety, and $Z := X^{\mathbb{N}}$. Then every chain $Z_1 \supseteq Z_2 \supseteq \ldots$ of **Sym** $\times$ **GL**-stable closed subvarieties of Z is eventually constant.

Taking X with trivial **GL**-action “$\Rightarrow$” Cohen’s theorem.

Taking $Z_i = X_i^{\mathbb{N}} \Rightarrow$ Noetherianity of **GL**-varieties.

A necessary generalisation

Example
Take $X = \mathbf{A}^{[(2),(1,1)]} = \{\mathbb{N} \times \mathbb{N}\text{-matrices}\}$, $Z = X^{\mathbb{N}}$, and $Z' \subseteq Z$ defined by the **Sym** $\times$ **GL**-orbit of the equation $(x_{1,(12)} - x_{1,(21)}) \cdot (x_{1,(12)} + x_{1,(21)}) = 0$.

So $Z'(K) = \{(M_1, M_2, \ldots) \mid \text{each } M_i \text{ is symmetric or skew}\}$.

We think of Z' as "covered" by the "smaller" $\mathbf{Sym}^2 \times \mathbf{GL}$-variety $(\mathbf{A}^{(2)})^{\mathbb{N}} \times (\mathbf{A}^{(1,1)})^{\mathbb{N}}$.

Theorem [Chiu-Danelon-D-Eggermont-Farooq]
Let $X_1, \ldots, X_k$ be a **GL**-varieties, and $Z := X_1^{\mathbb{N}} \times \cdots \times X_k^{\mathbb{N}}$. Then every chain $Z_1 \supseteq Z_2 \supseteq \ldots$ of $\mathbf{Sym}^k \times \mathbf{GL}$-stable closed subvarieties of Z is eventually constant.

Induction via product type

Setting
Y a **GL**-variety, and for $i = 1, \ldots, k$ a **GL**-subvariety $B \subseteq Y \times \mathbb{A}^{n_i}$ and a tuple $\underline{\lambda}_i$ of partitions. Then the $\mathbf{Sym}^k \times \mathbf{GL}$-variety $\prod_{i=1}^{k}(B_i \times \mathbf{A}^{\underline{\lambda}_i})^{\mathbb{N}}$ (product over Y) is *of product type*.

Key technical result:

Theorem [CDDEF]
A proper closed $\mathbf{Sym}^k \times \mathbf{GL}$-stable subvariety of a product-type variety can be covered by finitely many product-type varieties that are *lexicographically smaller*.

E.g. $B \subseteq Y \times \mathbb{A}^n, B' \subseteq Y' \times \mathbb{A}^{n'} \rightsquigarrow B \times \mathbf{A}^{\underline{\lambda}} \preceq B' \times \mathbf{A}^{\underline{\lambda}'}$ iff $\underline{\lambda}$ is strictly contained in a shift of $\underline{\lambda}'$ or else $\underline{\lambda} = \underline{\lambda}'$, $n = n'$, and $\mathrm{LM}(B) = \{\text{lms of monic poly in ideal of } B\} \supseteq \mathrm{LM}(B')$.

Open questions

1. Cohen's theorem fails for multiple unbounded indices. Is there still a coarse structure theory for **Sym**-stable varieties of matrices or tensors?

2. Does Noetherianity for **GL**-varieties hold at the scheme-theoretic level? (**No** in characteristic p by [Ganapathy]!)

3. Do our results for $\mathbf{Sym}^k \times \mathbf{GL}$-varieties hold in characteristic p? (The structure theory for **GL**-varieties was developed by [Bik-D-Snowden].)

4. What are the $\mathbf{Sym} \times \mathbf{GL}$-results good for? (E.g. there is an injection from $\{$representable matroids$\}$ to $\{\mathbf{Sym} \times \mathbf{GL}$-stable subvarieties of $(\mathbf{A}^{[(1)]})^{\mathbb{N}}\}$; but Noetherianity does not seem to imply nice properties for matroids.)

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