The algebra of symmetric high-dimensional data

Jan Draisma
TU Eindhoven

Kolloquium Köln, 2014

The algebra of symmetric high-dimensional data

·	3	0	3	1	3	0	3	6	·
·	6	0	6	2	6	0	1	2	·

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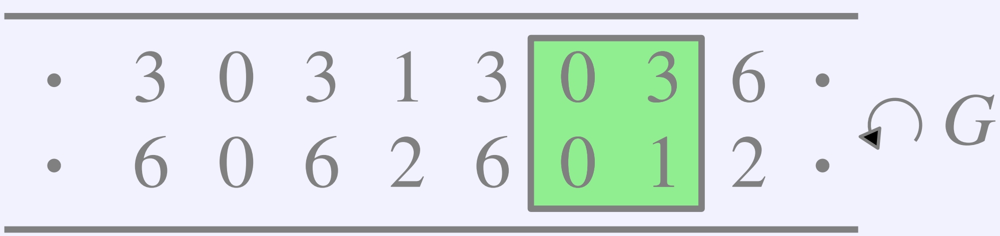
What global data properties can be tested locally?



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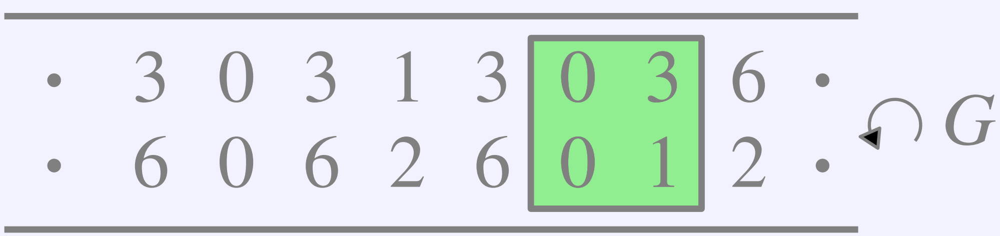

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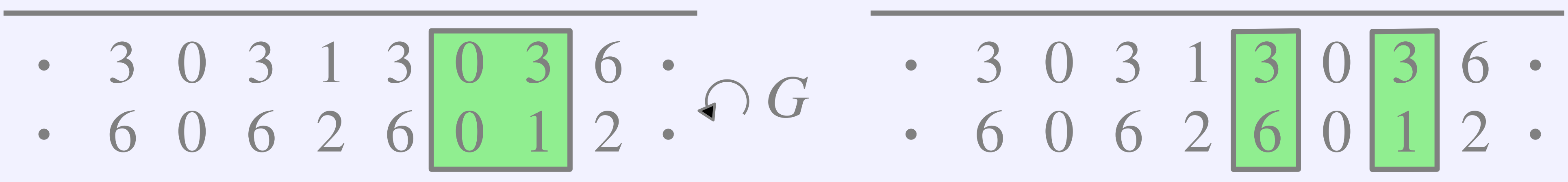

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The algebra of symmetric high-dimensional data

alg statistics | multilin alg | alg geometry | combinatorics

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↻ G

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History: Hilbert's Basis Theorem

David Hilbert

Any polynomial system

$f_1(x_1, \ldots, x_n) = 0,$

$f_2(x_1, \ldots, x_n) = 0, \ldots$

reduces to a *finite* system

($\rightsquigarrow$ *Noetherianity* of $K[x_1, \ldots, x_n]$)

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Das ist nicht Mathematik,
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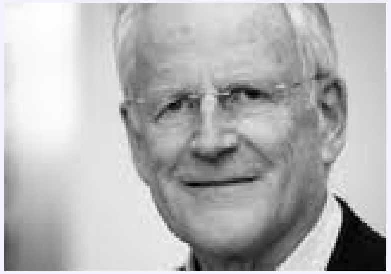
Bruno Buchberger

Gröbner bases, algorithmic methods



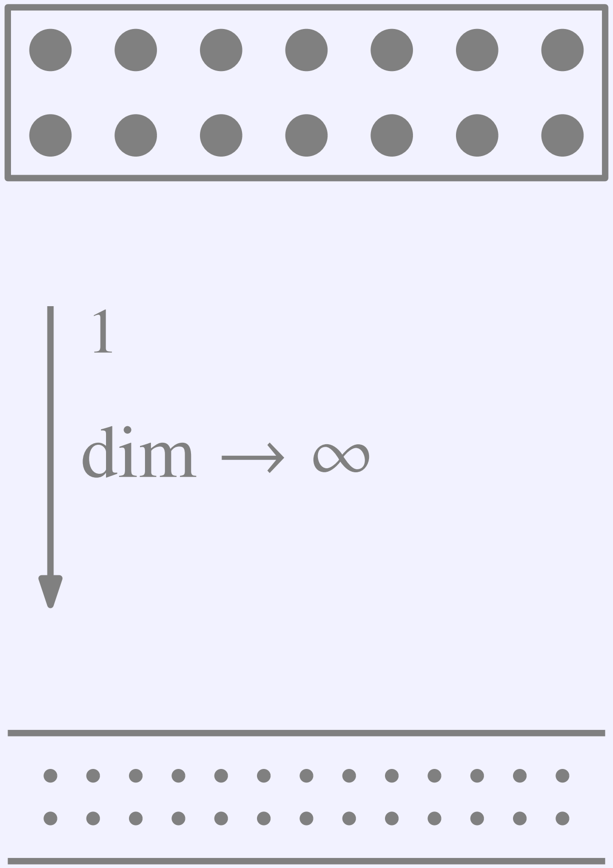
1. Model

high-dim data ⇝ ∞-dim data space

data property ⇝ ∞-dim subvariety

small window ⇝ finite window

⇝ *leave fin-dim commutative algebra*



Three-stage approach

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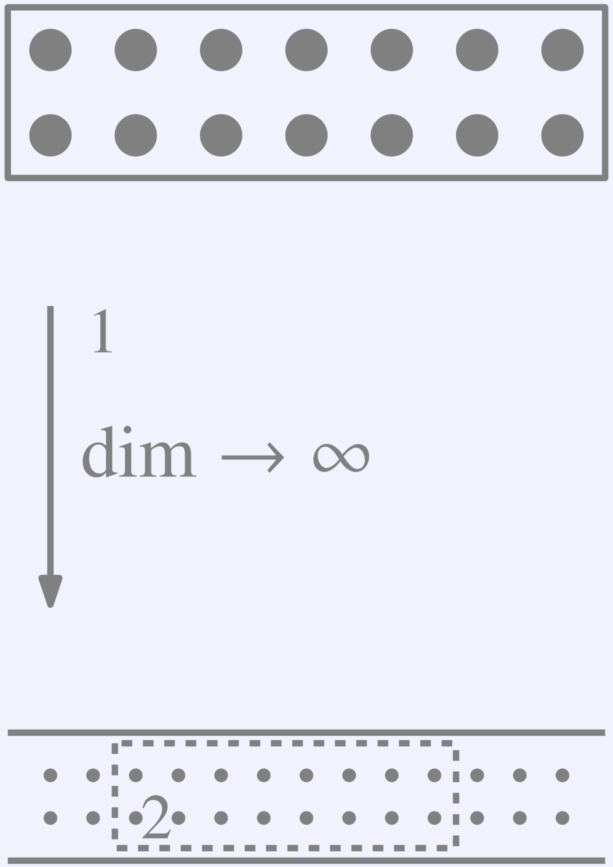
∞-dim property finitely defined

up to symmetry?

⇝ *generalise Basis Theorem to ∞ variables*



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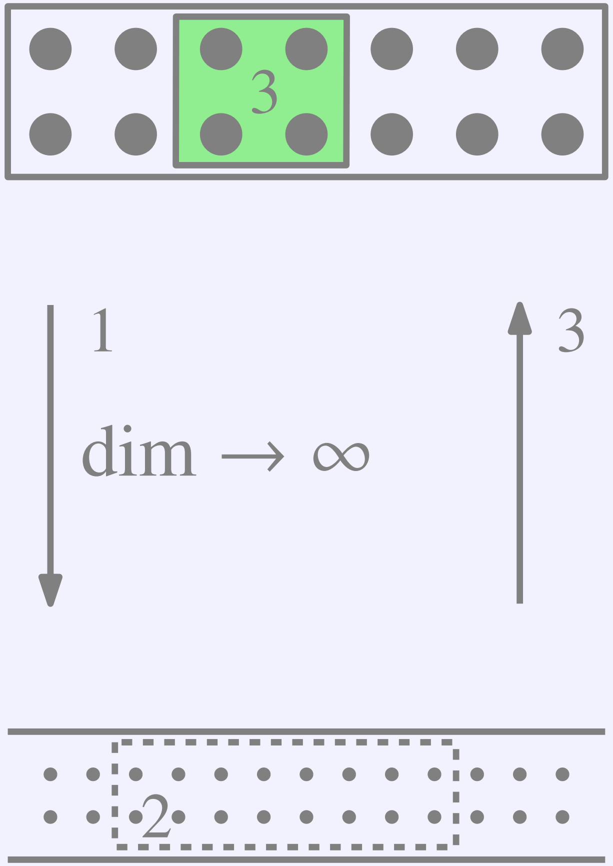
3. Compute

actual windows for fin-dim data
⇝ *generalise Buchberger alg to ∞ variables*



Noetherianity up to symmetry

Example

$K[x_1, x_2, \ldots]$ is not Noetherian, e.g. $x_1 = 0, x_2 = 0, \ldots$ does not reduce to a finite system.

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$K[x_1, x_2, \ldots]$ *is* Sym($\mathbb{N}$)-Noetherian, i.e., every Sym($\mathbb{N}$)-stable system reduces to finitely many equations up to Sym($\mathbb{N}$).

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Notion of G-Noetherianity generalises to G-actions on rings or topological spaces.

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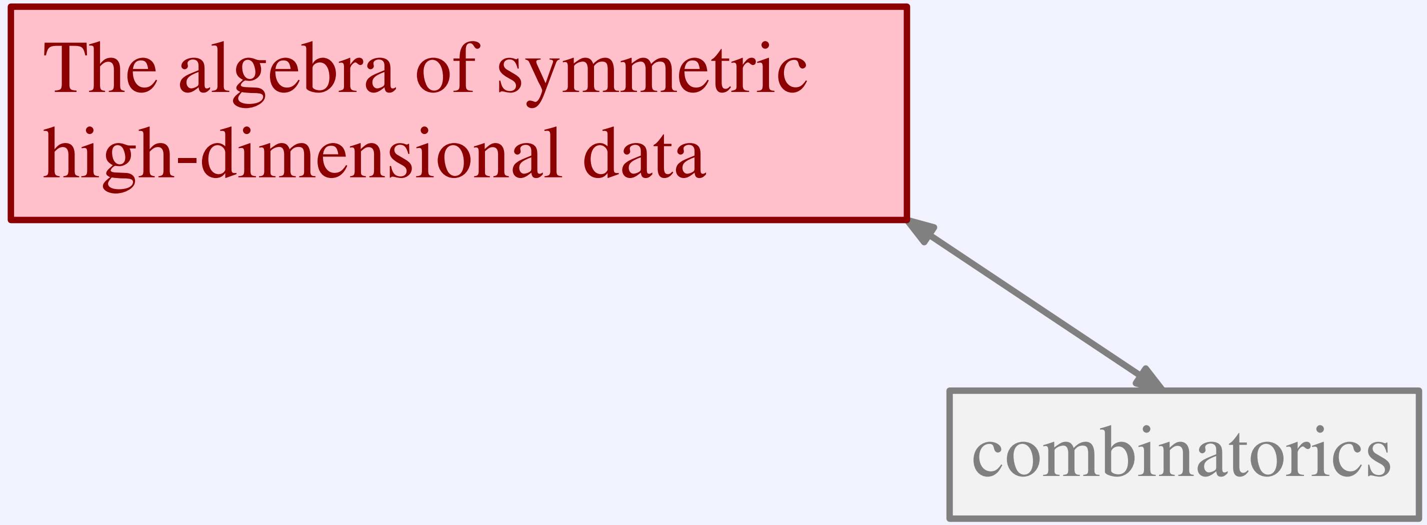
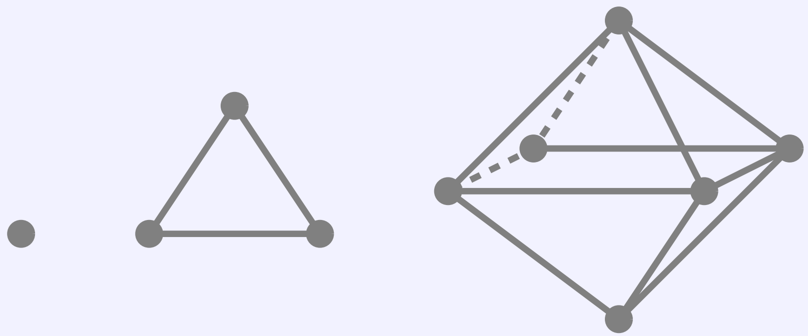
Fundamental (non-)examples

$K[x_{ij} \mid i \in \{1, \ldots, k\}, j \in \mathbb{N}]$ is Sym($\mathbb{N}$)-Noetherian;

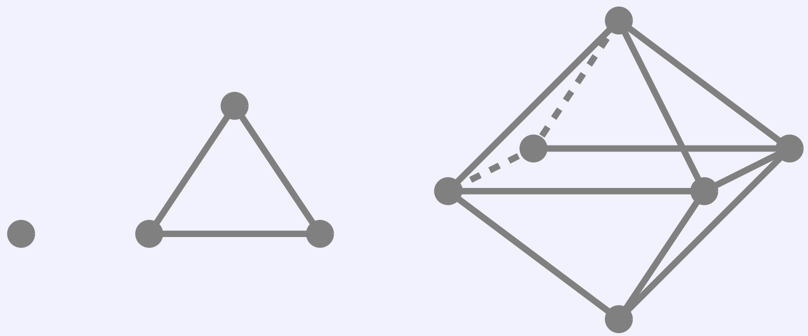
$K[x_{ij} \mid i, j \in \mathbb{N}]$ is *not* Sym($\mathbb{N}$)-Noetherian, *but* it *is* $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian, and so is $(K^{\mathbb{N}\times\mathbb{N}})^p$ for all p.


The algebra of symmetric high-dimensional data
combinatorics



Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



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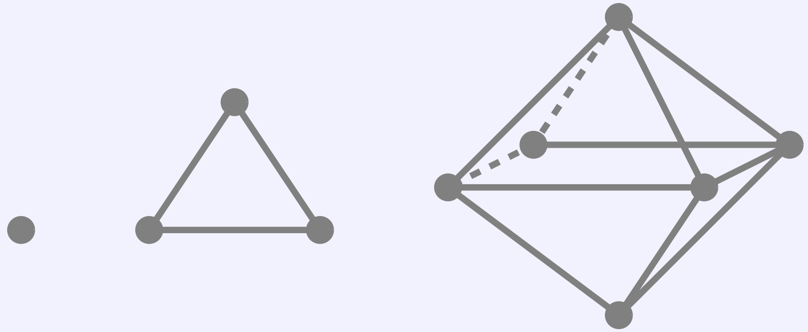
Markov basis M_n of integral relations

$v_{ij} = v_{ji}$ and $v_{ij} + v_{kl} = v_{il} + v_{kj}$ for i, j, k, l distinct
$\rightsquigarrow$ if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$,
then the expressions are connected by such
moves without creating negative coefficients



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Draisma-Eggermont-Krone-Leykin [*2013*]

For *any* family $(P_n \subseteq \mathbb{Z}^{k \times n})$, if $P_n = \mathrm{Sym}(n) P_{n_0}$ for $n \geq n_0$, then
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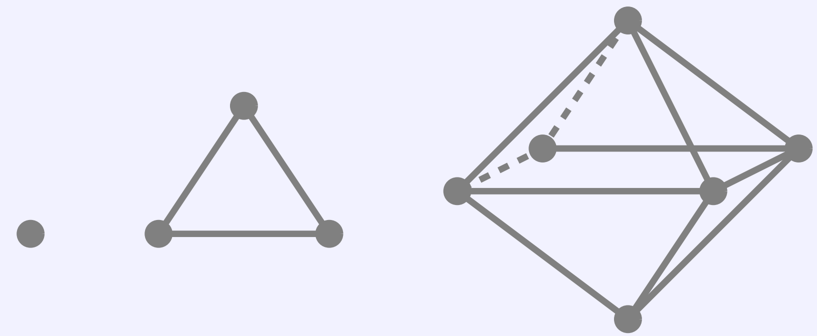

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$\rightsquigarrow$ *uses combinatorial notions such as well-partial orders*

$\rightsquigarrow$ *we also have an algorithm for computing n_1 and M_{n_1}*

The algebra of symmetric high-dimensional data

alg statistics

Gaussian factor analysis

Setting

$X_1, \ldots, X_n$ jointly Gaussian, mean 0

$\rightsquigarrow$ explained well by $k \ll n$ *factors*?

i.e., is $X_i = \sum_j s_{ij} Z_j + t_i \epsilon_i$, with $Z_1, \ldots, Z_k$,

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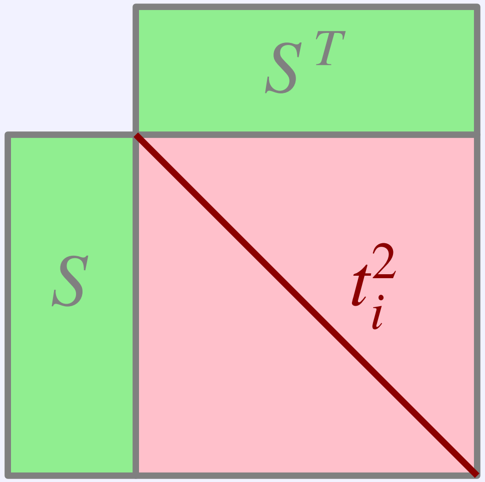

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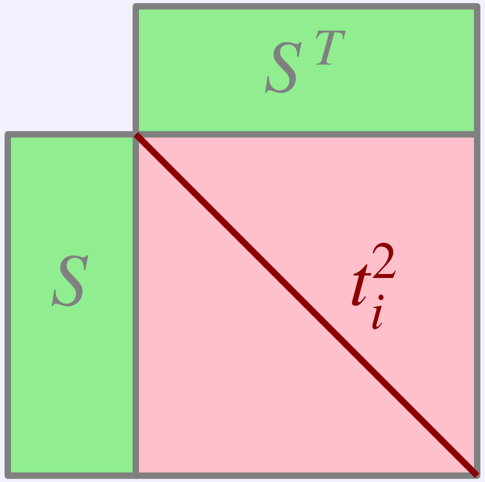

Definition

$$F_{k,n} := \overline{\{S S^T + \mathrm{diag}(t_1^2, \dots, t_n^2) \mid S \in \mathbb{R}^{n \times k}, t_i \in \mathbb{R}\}}$$

$\rightsquigarrow$ *algebraic variety in* $\mathbb{R}^{n \times n}$ *called Gaussian k-factor model*

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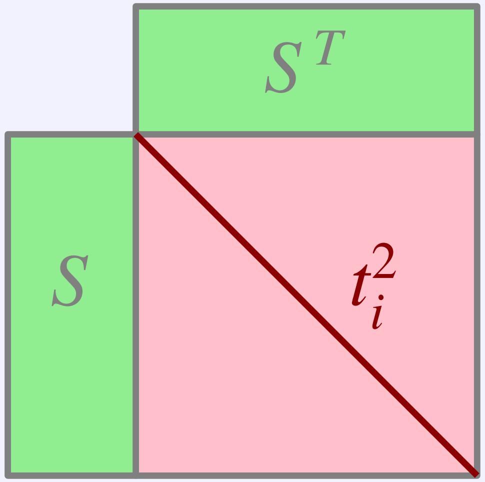

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Example

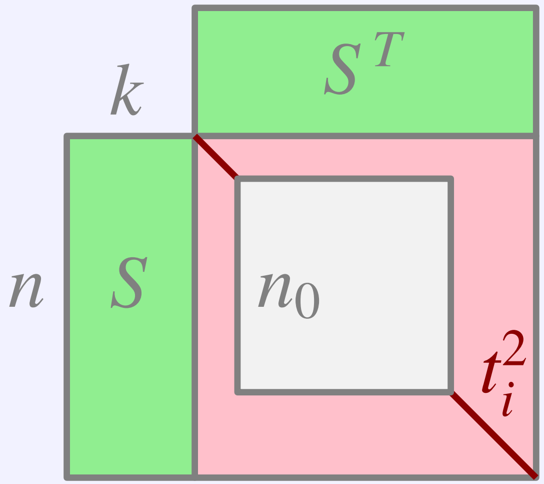
$F_{2,5}$ is zero set of $\{\sigma_{ij} - \sigma_{ji} \mid i, j = 1, \ldots, 5\}$ and the *pentad*
$\sum_{\pi \in \mathrm{Sym}(5)} \mathrm{sgn}(\pi) \sigma_{\pi(1)\pi(2)} \sigma_{\pi(2)\pi(3)} \sigma_{\pi(3)\pi(4)} \sigma_{\pi(4)\pi(5)} \sigma_{\pi(5)\pi(1)}$

Finiteness for the Gaussian k-factor model

Drton-Sturmfels-Sullivant [*Prob Th Rel Fields 2007*]

If $\Sigma \in F_{k,n}$ then any principal $n_0 \times n_0$ submatrix $\Sigma' \in F_{k,n_0}$.

$\rightsquigarrow$ *Is there an* $n_0 = n_0(k)$ *such that the converse holds for* $n \geq n_0$*?*



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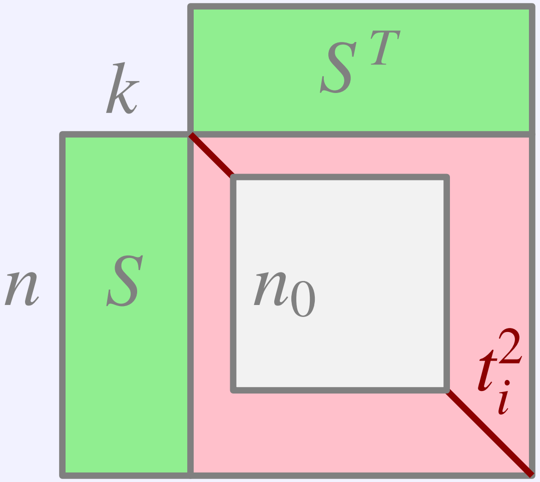

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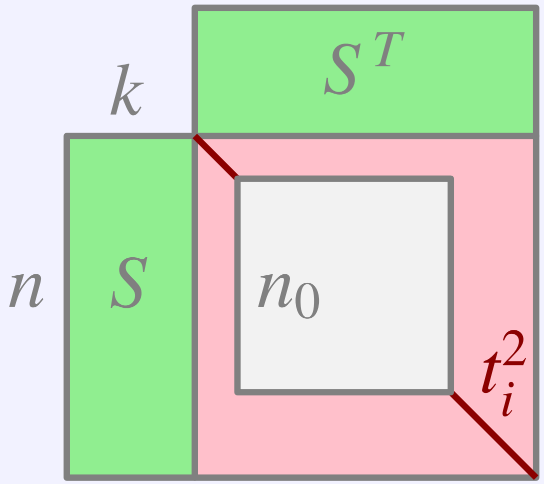
$\rightsquigarrow$ *uses* $F_{k,\infty}$ *and Noetherianity up to* $\mathrm{Sym}(\mathbb{N})$



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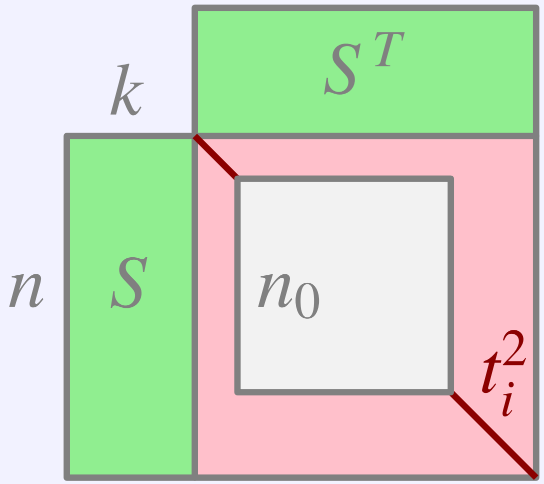

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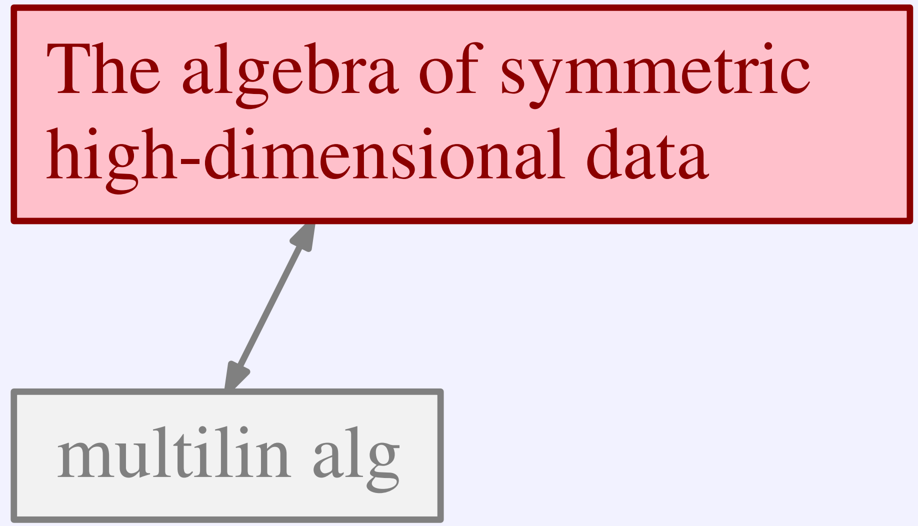
Brouwer-Draisma [*Math Comp 2011*]

yes for $k = 2$: pentads and 3×3-minors define $F_{2,n}$, $n \geq n_0 := 6$

$\rightsquigarrow$ *uses* Sym($\mathbb{N}$)*-Buchberger algorithm (+ a weekend on 20 computers)*

$\rightsquigarrow$ *a* **single** *computation proves this* **for all** n


The algebra of symmetric high-dimensional data
multilin alg

A wrong-titled movie

tensor T=multi-indexed array of numbers

matrices=two-way tensors

this picture=three-way tensor, $\ldots$

A wrong-titled movie

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this picture=three-way tensor, ...

Pure tensor P

has entries $P_{i,j,\ldots,k} = x_i y_j \cdots z_k$

for vectors $x, \ldots, z$

$\rightsquigarrow$ *for a matrix:* xy^T, *rank one*

Tensor rank

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Tensor rank of T

is minimal k in $T = \sum_{j=1}^{k} P^{(j)}$ with each $P^{(j)}$ pure

⇝ *generalises matrix rank*

⇝ *useful for MRI data, communication complexity, phylogenetics etc.*

Finiteness for bounded-rank tensors

Matrix rank	**Tensor rank**
efficiently computable	NP-hard
field independent	field dependent
can only go down in limit	can also go up

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Border rank of T

is smallest rank of T' arbitrarily close to T

$\rightsquigarrow$ *also extremely useful*

$\rightsquigarrow$ *for matrices coincides with rank*

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Draisma-Kuttler [*Duke 2014*]

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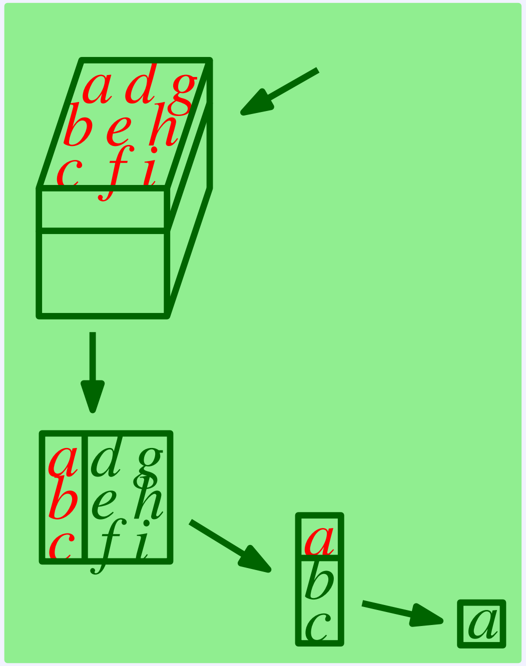

$\bigcup_{n=0}^{\infty} \mathrm{Sym}(n) \ltimes \mathrm{GL}_3^n$

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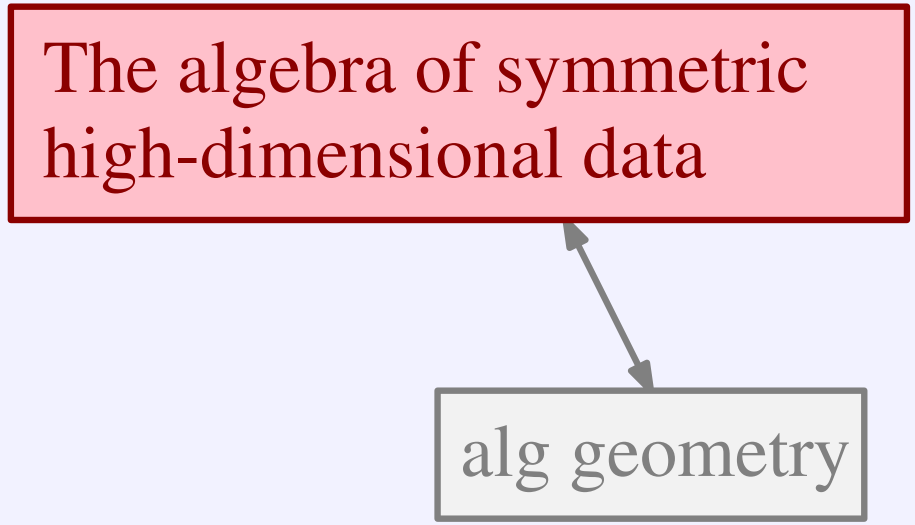
The algebra of symmetric high-dimensional data
alg geometry

Grassmannians: functoriality and duality

V a fin-dim vector space over an infinite field K
$\rightsquigarrow \mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
cone over Grassmannian
(*rank-one alternating tensors*)

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Two properties:

1. if $\varphi : V \to W$ linear
$\rightsquigarrow \bigwedge^p \varphi : \bigwedge^p V \to \bigwedge^p W$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_p(W)$

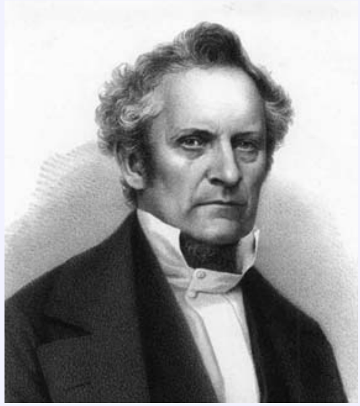
Grassmannians: functoriality and duality

V a fin-dim vector space over an infinite field K
$\rightsquigarrow \mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
cone over Grassmannian
(*rank-one alternating tensors*)

Two properties:

1. if $\varphi : V \to W$ linear
$\rightsquigarrow \bigwedge^p \varphi : \bigwedge^p V \to \bigwedge^p W$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_p(W)$

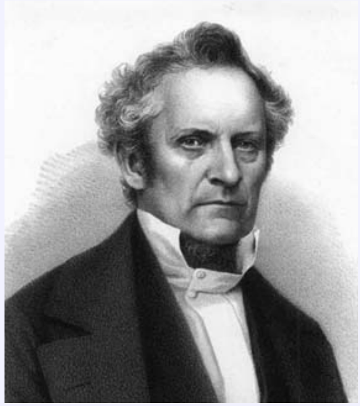
2. if $\dim V =: n + p$ with $n, p \geq 0$
$\rightsquigarrow$ natural map $\bigwedge^p V \to (\bigwedge^n V)^* \to \bigwedge^n(V^*)$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_n(V^*)$



Definition

Rules $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ with

$$\mathbf{X}_p : \{\text{vector spaces } V\} \to \{\text{varieties in } \bigwedge^p V\}$$



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form a *Plücker variety* if, for $\dim V = n + p$,

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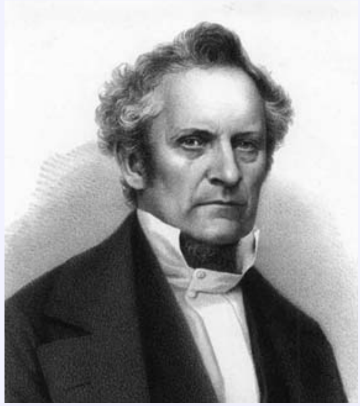

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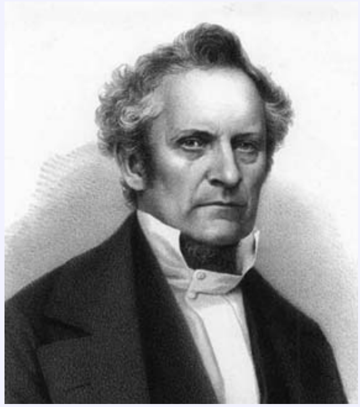
Constructions

$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are

$\mathbf{X} + \mathbf{Y}$ *(join)*, $\tau\mathbf{X}$ *(tangential)*,

$\mathbf{X} \cup \mathbf{Y}, \mathbf{X} \cap \mathbf{Y}$

Plücker varieties



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skew analogue of Snowden's Δ-varieties

Definition

A Plücker variety $\{\mathbf{X}_p\}_p$ is *bounded* if $\mathbf{X}_2(V) \neq \bigwedge^2 V$ for $\dim V$ sufficiently large.

Finiteness for Plücker varieties, with Eggermont

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Theorems apply, in particular, to
$k\mathbf{Gr}$ = k-th *secant variety* of $\mathbf{Gr}$.

The infinite wedge

$$V_\infty := \langle \dots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \dots \rangle_K$$
$$V_{n,p} := \langle x_{-n}, \dots, x_{-1}, x_1, \dots, x_p \rangle \subseteq V_\infty$$

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Diagram

$$\begin{array}{ccccccc}
\bigwedge^0 V_{00} & & \bigwedge^1 V_{01} & & \bigwedge^2 V_{02} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
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\end{array}$$

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Definition

$\bigwedge^{\infty/2} V_\infty := \lim_{\rightarrow} \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \ldots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

The limit of a Plücker variety

Dual diagram

$$\begin{array}{ccccc}
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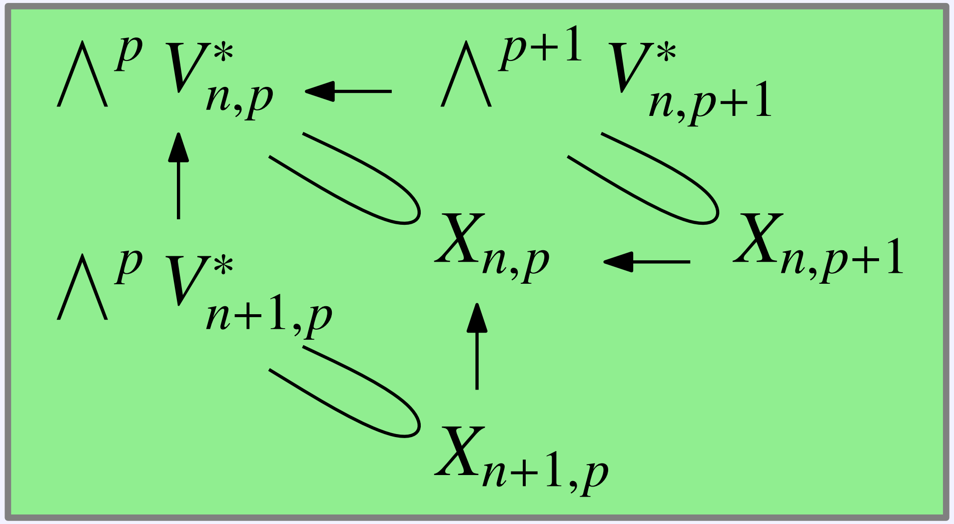

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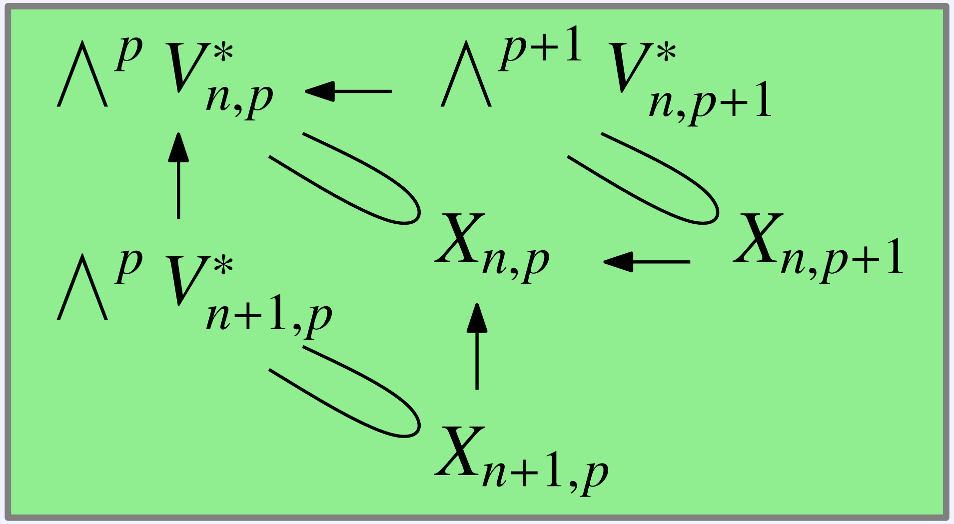

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Theorem (implies earlier)
For *bounded* $\mathbf{X}$, the limit $\mathbf{X}_\infty$ is cut out by finitely many GL_∞-orbits of equations.

Example

The limit $\mathbf{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$ of $(\mathbf{Gr}_p)_p$ is *Sato's Grassmannian* defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$ where $i_k = k - 1$ for $k \gg 0$ and $j_k = k + 1$ for $k \gg 0$.

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Sato's Grassmannian

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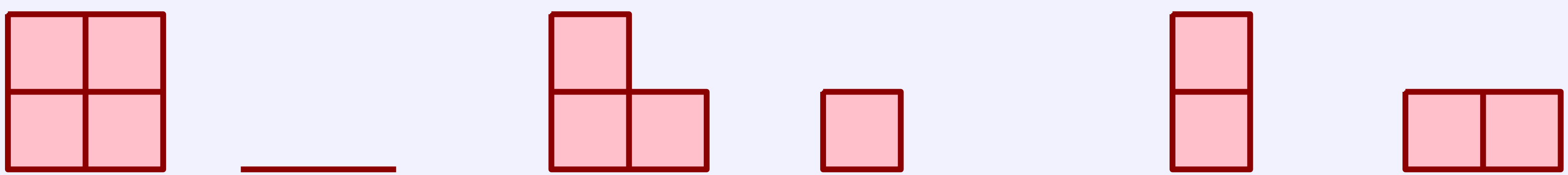
But in fact the GL_∞-orbit of

$(x_{-2,-1,3,\ldots} \cdot x_{1,2,3,\ldots}) - (x_{-2,1,3,\ldots} \cdot x_{-1,2,3,\ldots}) + (x_{-2,2,3,\ldots} \cdot x_{-1,1,3,\ldots})$



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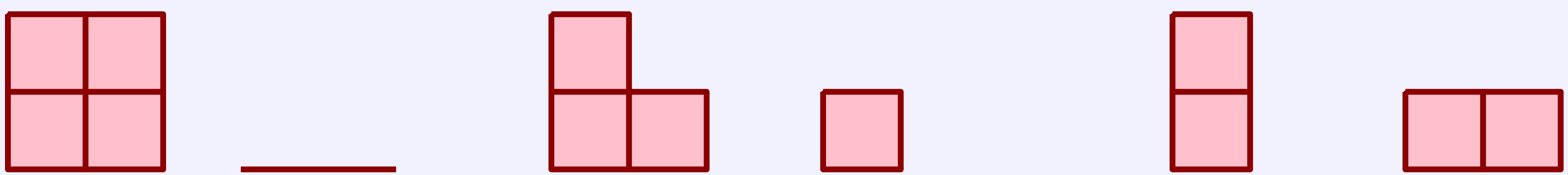

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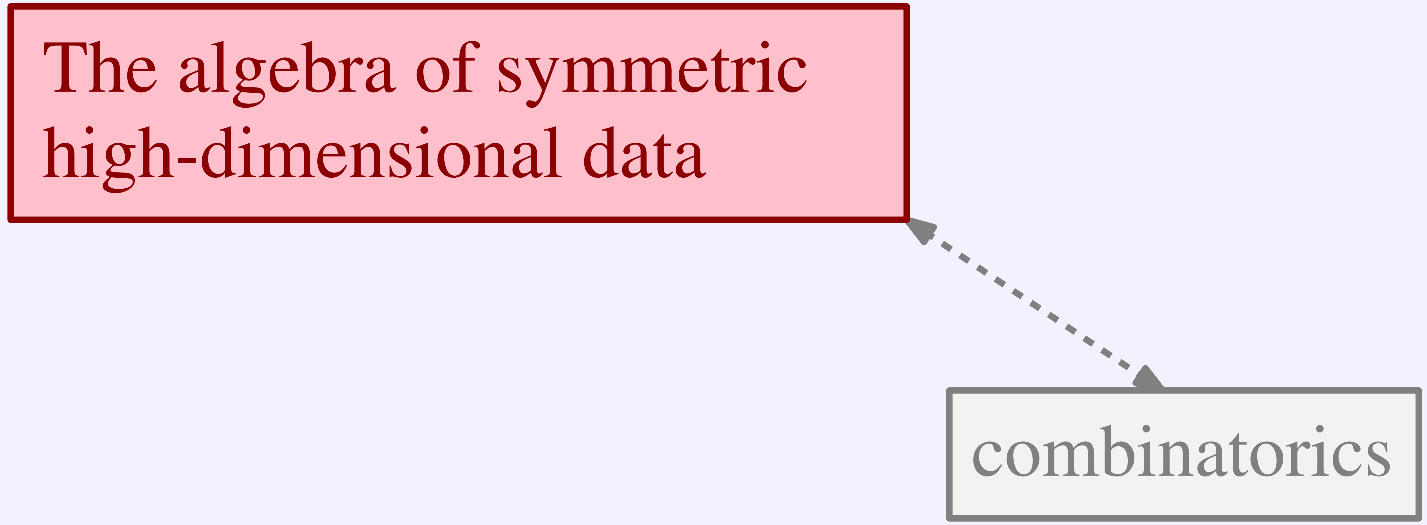
Our theorems imply that also higher secant varieties of Sato's Grassmannian are defined by finitely many GL_∞-orbits of equations. . . *we just don't know which!*

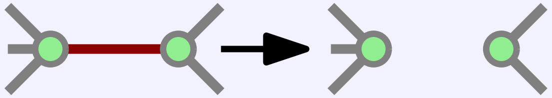
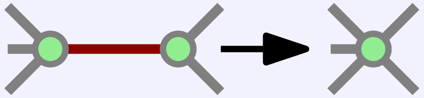


Conjecture

Over any field K, Sato's Grassmannian $\mathbf{Gr}_\infty(K)$ is Noetherian up to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N}) \subseteq \mathrm{GL}_\infty$.

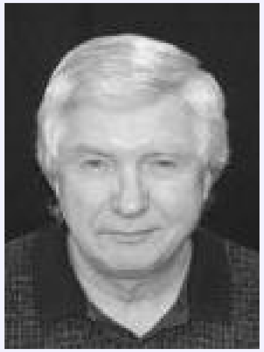
Graph minors

Any sequence of operations

 and 

takes a graph to a *minor*.

The graph minor theorem



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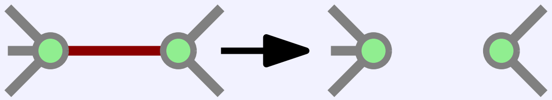
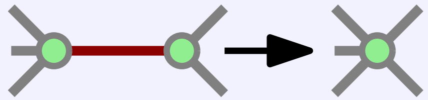
 and 

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Robertson-Seymour [JCB 1983–2004, 669pp]

Any network property preserved under taking minors can be characterised by *finitely many forbidden minors*.

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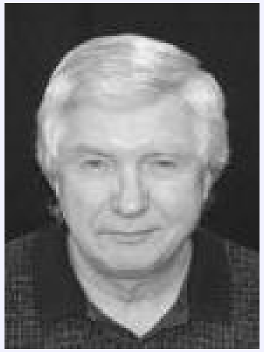

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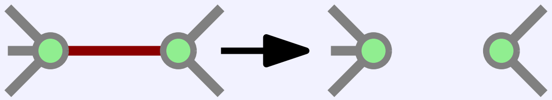
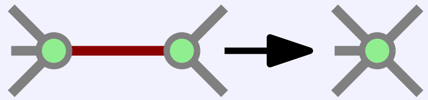
 and 

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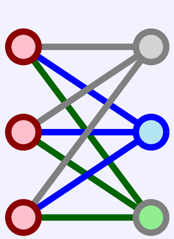
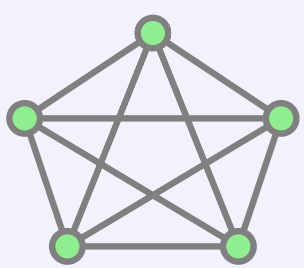
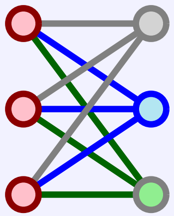
Wagner [Math Ann 1937]

For *planarity* these are

 and 

From graphs to matroids



1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
−1	−1	−1	0	0	0	0	0	0
0	0	0	−1	−1	−1	0	0	0
0	0	0	0	0	0	−1	−1	−1

column independence structure = matroid

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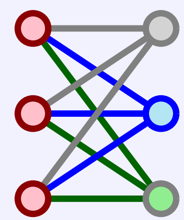

1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
−1	−1	−1	0	0	0	0	0	0
0	0	0	−1	−1	−1	0	0	0
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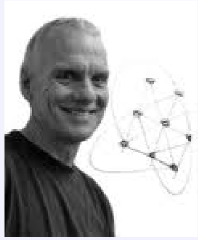
Matroid minor theorem (Geelen-Gerards-Whittle)
Any minor-preserved property of matroids over a fixed *finite field K* can be characterised by finitely many forbidden minors.



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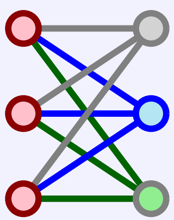

1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
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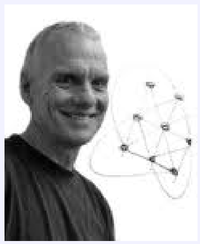

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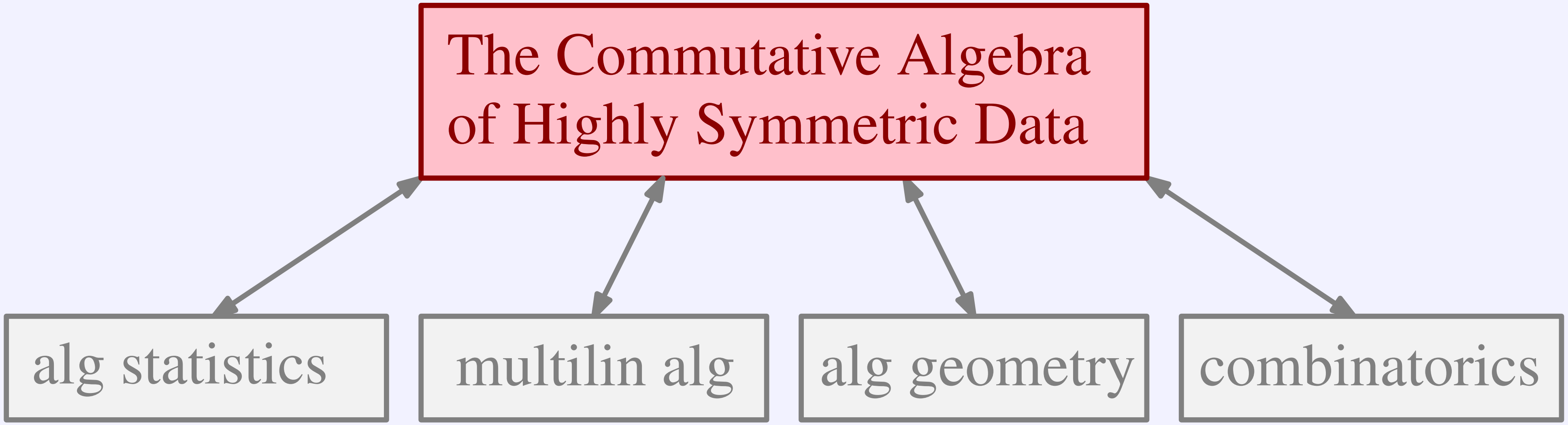
Any minor-preserved property of matroids over a fixed *finite field K* can be characterised by finitely many forbidden minors.

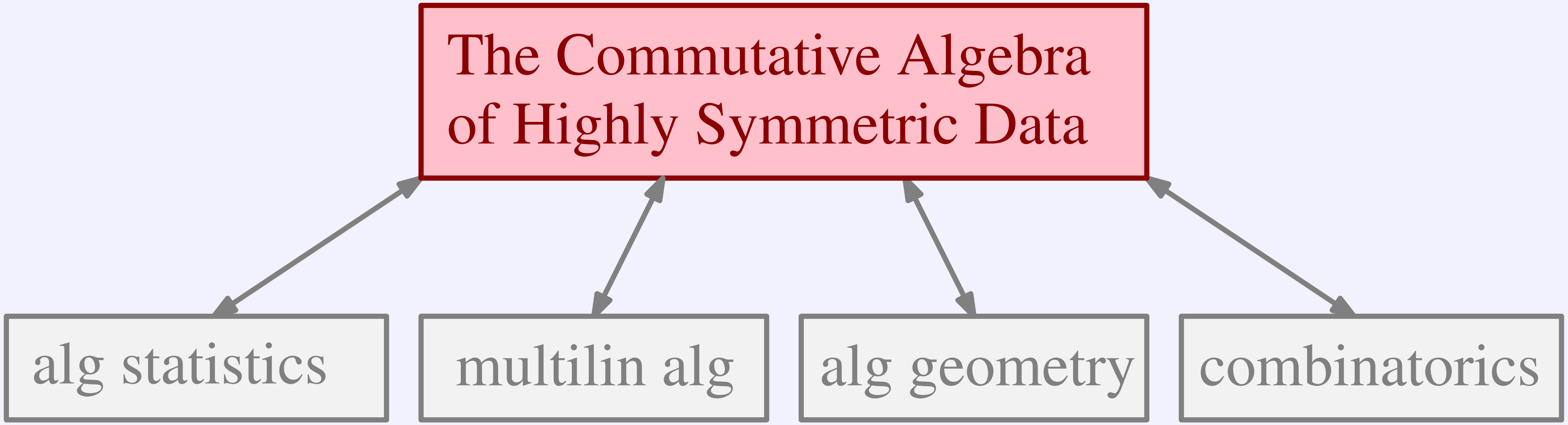
Correspondence

"Equivalant" to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N})$-Noetherianity of $\mathbf{Gr}_\infty(K)$ (but Noetherianity may hold even for infinite K).



⇝ *theory and algorithms for highly symmetric, ∞-dim varieties*

⇝ *exciting interplay of algebra, combinatorics, statistics, and geometry*



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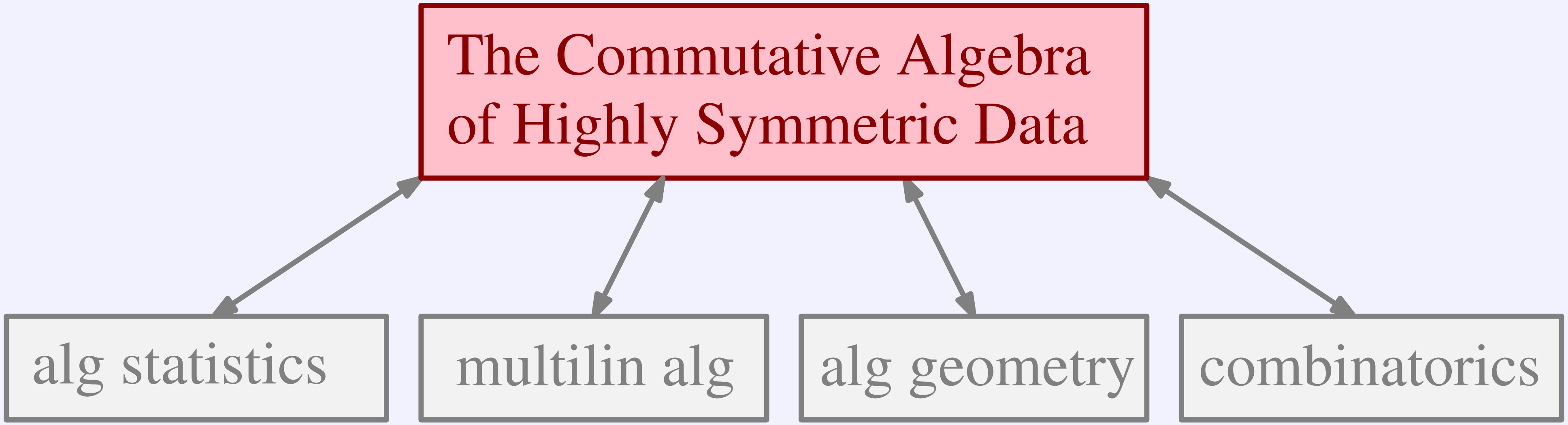

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Ich habe mich davon überzeugt, daß
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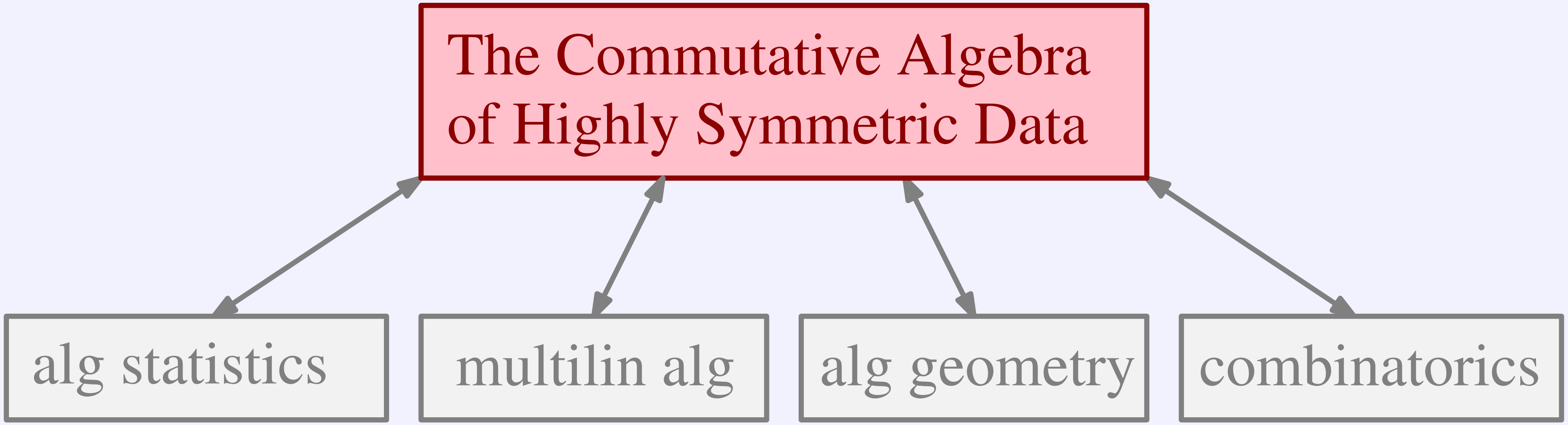

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Thank you!