Set-theoretic finiteness for the k-factor model

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The main theorem

M_n: $n \times n$-matrices
$\mathrm{OM}_n \cong \mathbb{A}^{n^2-n}$: off-diagonal $n \times n$-matrices
$\mathrm{M}_n^{\leq k}$: of rank $\leq k$
$\mathrm{OM}_n^{\leq k}$: image closure of $\mathrm{M}_n^{\leq k}$
K: a field

Observation. *For k fixed and $n \geq 2(k+1)$:*

$$\mathrm{M}_n^{\leq k}(K) = \{y \in \mathrm{M}_n(K) \mid \forall I, |I| = 2(k+1) : y[I] \in \mathrm{OM}_{2(k+1)}^{\leq k}(K)\}.$$

Theorem. *For k fixed there exists an $n_0 = n_0(k)$ such that for $n \geq n_0$:*

$$\mathrm{OM}_n^{\leq k}(K) = \{y \in \mathrm{OM}_n(K) \mid \forall I, |I| = n_0 : y[I] \in \mathrm{OM}_{n_0}^{\leq k}(K)\}.$$

Theorem. *A similar statement holds for* symmetric *matrices.*

Remarks

1. The proof is not constructive.
2. For $k = 1$ $n_0 = 4$ suffices (toric ideal).
3. For $k = 2$ we think $n_0 = 6$ suffices
   (symmetric case: Drton and student, Very Recently)
4. The statement is just *set-theoretical*.
5. Drton-Sturmfels-Sullivant raised this question (2007).

Example (Pentad for symmetric case (Kelly, 1935)).
$k = 2$ and $n = 5$
$\dim \mathrm{SOM}_5 = \binom{5}{2} = 10$
$\dim \mathrm{SOM}_5^{\leq 2} = 9$
hyperplane with equation

$$\sum_{\pi} \mathrm{sgn}\,(\pi) y_{\pi(1),\pi(2)} y_{\pi(2)\pi(3)} y_{\pi(3),\pi(4)} y_{\pi(4)\pi(5)} y_{\pi(5),\pi(1)} = 0$$

Motivation: model selection

Gaussian distribution on $n+k$ variables $Z_1, \ldots, Z_{n+k}$:

$$f_Z(z) = \frac{1}{(2\pi)^{n/2}\det(A)^{1/2}} \exp(-\frac{1}{2} z^T A^{-1} z)$$

with covariance matrix $A > 0$ and mean 0
$i, j \in I := \{1, \ldots, n\}$ and $J := \{n+1, \ldots, n+k\}$
$Z_i \perp Z_j | \{Z_{n+1}, \ldots, Z_{n+k}\}$ iff
$\det \begin{bmatrix} A[i,j] & A[i,J] \\ A[J,j] & A[J] \end{bmatrix} = 0$
and for all i, j iff
$A[I] - A[I,J]A[J]^{-1}A[I,J]^T$ is diagonal
Parameter space for the Gaussian k-factor model on n observed variables is $\{D + S \mid D$ *diagonal* > 0 *and* $S > 0$ *rank* $\leq k\}$, *a semi-algebraic set.*

Application: 7 or 9 types of intelligence? (Howard Gardner)

A reformulation

$\mathrm{OM}_\infty := \lim_{\leftarrow} \mathrm{OM}_n$
coordinate ring: $K[y_{ij} \mid i, j \in \mathbb{N}, i \neq j]$
$\mathrm{OM}_\infty^{\leq k} := \lim_{\leftarrow} \mathrm{OM}_n^{\leq k}$

Theorem. *For fixed k, there exist finitely many polynomials $f_1, \ldots, f_l \in K[y_{ij}]$ such that*

$$\mathrm{OM}_\infty^{\leq k}(K) = \{y \in \mathrm{OM}_\infty(K) \mid f_i(gy) = 0 \textit{ for all } g \in \mathrm{Sym}(\mathbb{N})\}$$

Remark. Actually, any $\mathrm{Sym}(\mathbb{N})$-stable subvariety is finitely defined in this sense.

Ring-theoretic G-Noetherianity

R ring
G group acting on R

Definition. R is *G-Noetherian* if every ascending chain of G-stable ideals stabilises.

Theorem (Aschenbrenner-Hillar, 2007). *$R = K[x_1, x_2, \ldots]$ is $G = \mathrm{Sym}(\mathbb{N})$-Noetherian*

Proof: define a suitable partial order on monomials and prove that it is a well-quasiorder, as well as compatible with Groebner-basis type arguments.

Theorem (Hillar-Sullivant, 2007). *$K[x_{i,1}, x_{i,2}, \ldots \mid i = 1, \ldots, l]$ is $\mathrm{Sym}(\mathbb{N})$-Noetherian.*

But $K[y_{i,j} \mid i \neq j]$ is *not* $\mathrm{Sym}(\mathbb{N})$-Noetherian!

Lemma (Hilbert). *R G-Noetherian implies $R[X]$ G-Noetherian.*

Topological G-Noetherianity

X topological space
G group acting on X

Definition. X is *G-Noetherian* if every descending chain of G-stable closed subsets stabilises.

Lemma. 1. *X G-Noetherian $\Rightarrow$ every G-stable closed subset of X G-Noetherian.*

2. *$X \dot{\cup} Y$ is G-Noetherian iff X and Y are.*

3. *X G-Noetherian, $f : X \to Y$ surjective and G-Noetherian $\Rightarrow$ Y G-Noetherian.*

Proposition. *$H \subseteq G$ and X is H-Noetherian $\Rightarrow$ $G \times_H X$ is G-Noetherian.*

A stronger result

$\tilde{\mathrm{OM}}_\infty^{\leq k} \subseteq \mathrm{OM}_n$ defined by the off-diagonal $(k+1) \times (k+1)$-minors

Theorem. $\tilde{\mathrm{OM}}_\infty^{\leq k}(K)$ *is* $\mathrm{Sym}(\mathbb{N})$-*Noetherian.*

This implies the earlier results: finitely many equations are needed to cut out $\tilde{\mathrm{OM}}_\infty^{\leq k}(K)$, and finitely many to cut out $\mathrm{OM}_\infty^{\leq k}(K)$ in $\tilde{\mathrm{OM}}_\infty^{\leq k}(K)$ by the theorem.

Proof sketch

Induction on k:

1. $\tilde{\mathrm{OM}}_\infty^{\leq 0}(K)$ is a single point
2. Assume $\tilde{\mathrm{OM}}_\infty^{\leq k-1}(K)$ is $\mathrm{Sym}(\mathbb{N})$-Noetherian.
   Write $\tilde{\mathrm{OM}}_\infty^{\leq k}(K) = \tilde{\mathrm{OM}}_\infty^{\leq k-1}(K) \cup Z$ where Z is the image of $\mathrm{Sym}(\mathbb{N}) \times_H X$ under some $\mathrm{Sym}(\mathbb{N})$-equivariant map
   $H := \mathrm{Sym}(\{2k+1, 2k+2, \ldots\})$
   X some space which is $\mathrm{Sym}(H)$-Noetherian by Hillar-Sullivant and Hilbert.

Construction of Z

Recall: Z contains all elements of $\tilde{\mathrm{OM}}_\infty^{\leq k}$ having some invertible off-diagonal $k \times k$-minor
$I := \{1, \ldots, k\}$, $J := \{k+1, \ldots, 2k\}$
$B \in K^{(\mathbb{N} \setminus J) \times [k]}$, $C \in K^{[k] \times (\mathbb{N} \setminus I)}$, $D \in K^{\mathbb{N} \times I}$, $E \in K^{J \times (\mathbb{N} \times I)}$

Now

$$\begin{bmatrix} D[I,I] & (B.C)[I,J] & (B.C)[I, \mathbb{N} \setminus (I \cup J)] \\ D[J,I] & E[J,J] & E[J, \mathbb{N} \setminus (I \cup J)] \\ D[\mathbb{N} \setminus (I \cup J), I] & (B.C)[\mathbb{N} \setminus (I \cup J), J] & (B.C)[\mathbb{N} \setminus (I \cup J), \mathbb{N} \setminus (I \cup J)] \end{bmatrix}$$

is an $H = \mathrm{Sym}(\mathbb{N} \setminus (I \cup J))$-equivariant expression in B, C, D, E. Move non-zero $k \times k$-minor around with $\mathrm{Sym}(\mathbb{N})$.

Outlook

1. Scheme-theoretic?
2. Positive definite? Constructive?
3. Use of invariant theory?
4. Other statistical models?
5. Vandermonde varieties!