SCHRIJVER'S AND DERKSEN'S RESULTS ON AN INVERSE PROBLEM OF INVARIANT THEORY

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Let K be an algebraically closed field of characteristic zero, and let V be a finite-dimensional vector space over K. Define

$$T := T(V^*) \otimes T(V).$$

Here $T(V)$ and $T(V^*)$ are the tensor algebras over V and V^*, respectively, so that T is canonically the direct sum of all spaces

$$V_q^p := (V^*)^p \otimes V^q.$$

The group $G := \mathrm{GL}(V)$ acts naturally on T, hence so does any subgroup H of G. We want to characterise the subsets of T that are of the form T^H for certain H.

First we do this for T^G. We need some observations and notation:

(1) T is an associative algebra (in which elements from $T(V)$ commute with elements of $T(V^*)$), and the V_q^p define a bigrading on T.
(2) The space V_1^1 can be identified with $\mathrm{End}(V)$, and the G-action on V_1^1 then corresponds to conjugation. Let I denote the element of V_1^1 corresponding to the identity in $\mathrm{End}(V)$.
(3) For $i = 1, \ldots, p$ and $j = 1, \ldots, q$ let $\partial_j^i : V_q^p \to V_{q-1}^{p-1}$ be the contraction of the i-th copy of V^* and the j-th copy of V. Note that ∂_j^i is a G-equivariant map.

Then we have the following classical first fundamental theorem for $\mathrm{GL}(V)$.

Theorem 0.1 (Weyl)**.** *T^G is the smallest subalgebra of T which is bigraded (i.e., $T^G = \bigoplus_{p,q}(T^G \cap V_q^p)$), contains I and is closed under all contractions.*

Now let H be any subgroup of G. Then one readily verifies that T^H is a bigraded subalgebra of T containing I and closed under contractions. However, not every subalgebra A of T with these properties is of the form T^H.

Example 0.2. Suppose that $\dim V = 1$, say $V^* = Kx$ and $V = Ke$ with $\langle x, e\rangle = 1$, so that $T = K[x, e]$. Take $A = \bigoplus_{i \geq j} Kx^i e^j$. Then A is a bigraded subalgebra of T containing $I = xe$ and closed under contraction, but only $1 \in \mathrm{GL}_1$ stabilises x, so that A is not of the form T^H.

So to characterise the T^H we need further conditions. It seems very hard to do this without without further assumptions on H. A natural assumption in invariant theory is that H be compact, or linearly reductive.

For a *compact* approach, suppose that $K = \mathbb{C}$ and V carries a Hermitian inner product. Then the FFT as above is still valid for G replaced with $\mathrm{U}(V)$ (since this group is dense in $\mathrm{GL}(V)$). For a subgroup H of $\mathrm{U}(V)$, T^H has more structure,

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namely: T^H is closed under the natural map $* : V_q^p \to V_p^q$ induced by the Hermitian form.

Theorem 0.3 (Schrijver). *The map $H \to T^H$ maps the closed subgroups of $U(V)$ bijectively onto the bi-graded subalgebras of T that contain I and are closed under contraction and $*$.*

However, a *reductive* approach would be more satisfactory, since the introduction of a Hermitian form seemed somewhat arbitrary. Instead, let $\langle .,. \rangle$ be the natural bilinear form on T coming from the pairing $V^* \times V \to K$, and let H be a linearly reductive subgroup of G. If U, U' are finite-dimensional H-submodules of T, then the restriction of $\langle .,. \rangle$ to $U \times U'$ is an H-invariant map to the trivial representation. In particular, if U and U' are irreducible, the form is trivial on $U \times U'$ unless $(U')^* = U$. Write $V_q^p = (V_q^p)^H \oplus U$ and $V_p^q = (V_p^q)^H \oplus U'$ as H-modules. Then the observation before shows that the form is zero on $(V_q^p)^H \times U'$ and on $U \times (V_p^q)^H$. Since it is non-degenerate on $V_q^p \times V_p^q$, we find that it pairs $(V_q^p)^H$ and $(V_p^q)^H$ non-degenerately. Summarising: if H is linearly reductive, then the form is non-degenerate on T^H.

Theorem 0.4 (Derksen). *The map $H \mapsto T^H$ is a bijection between (Zariski-)closed reductive subgroups of $\mathrm{GL}(V)$ and contraction-closed, bigraded subalgebras of T that contain I and are non-degenerate relative to $\langle .,. \rangle$.*

The proof seems to need a detour, which Derksen had in fact already developed before learning Schrijver's theorem. As a motivation for this detour, observe the following: if H is a reductive subgroup of G, then G/H is an affine variety, and

$$(K[G/H] \otimes T)^G \to T^H, f \mapsto f(eH)$$

is an algebra isomorphism. (Note that an element of $K[G/H] \otimes T$ can be regarded as a T-valued function on G/H.) Thus find that T^H can be thought of as the set of G-equivariant T-valued regular functions on some affine scheme. It turns out that a wide class of algebras related to T can be thought of in this way. We need the following definitions.

Definition 0.5. For any commutative K-algebra R with 1, $A := R \otimes T$ is an associative algebra which inherits a bigrading $A_q^p := R \otimes V_q^p$, R-linear contractions $\partial_j^i : A_q^p \to A_{q-1}^{p-1}$ and an element $I = 1 \otimes I \in A_1^1$ from T.

For any such R, any bigraded subalgebra of $R \otimes T$ which contains I and is closed under contractions is called a *tensalgebra*. Tensalgebras form a category in which the morphisms $A \to B$ of are algebra homomorphisms that respect the grading, map I_A to I_B, and intertwine $(\partial_j^i)_A$ and $(\partial_j^i)_B$.

The second category that plays a role is the category of commutative G-algebras, i.e., commutative K-algebras R with 1 with a right G-action given by an algebra homomorphism $R \to R \otimes K[G]$. The morphisms are G-equivariant K-algebra homomorphisms.

Now Derksen defines two functors. First, to any commutative G-algebra R he associates $\Phi(R) := (R \otimes T)^G$. For the functor in the opposite direction, we need the following lemma.

Lemma 0.6. *For any tensalgebra A there exists a commutative algebra R with 1 and a tensalgebra homomorphism $\psi : A \to R \otimes T$ such that any tensalgebra* *homomorphism $A \to S \otimes T$ factorises into ψ followed by a unique homomorphism of the form $\phi \otimes \mathrm{id}_T$, where ϕ is a homomorphism $R \to S$.*

We will set $\Theta(A) := R$.

Proof. For any $p, q \in \mathbb{N}$ let $\{v_i\}$ be a basis of V_q^p. For any $a \in A_q^p$ introduce variables $x_{i,a}$ and make the *Ansatz*

$$\psi(a) = \sum_i x_{i,a} \otimes v_i.$$

Now factor out the relations in $K[(x_{i,a})_{p,q,a,i}]$ that make this ψ into a tensalgebra homomorphism, and the resulting algebra obviously has the required properties. □

It turns out that the R associated to A above carries a natural G-algebra structure, and one checks Θ is a functor from tensalgebras to commutative G-algebras.

Theorem 0.7 (Derksen)**.** *Θ and Φ are each other's inverses.*

Example 0.8.
(1) $\Theta(T^H) \cong K[G/H]$ for all closed reductive subgroups H of G.
(2) With A as in the earlier example, $\Theta(A)$ can be found as follows: A is generated by x and xe, so any morphism of tensalgebras $A \to R \otimes T$ is determined by the images of x and xe, which must be of the form $s \otimes x$ and $t \otimes xe$, respectively, by the bi-gradedness. Moreover, $xe = I$, so $t = 1$. Now there are no further restrictions on s: the map sending $x^i e^j$ to $s^{i-j} \otimes x^i e^j$ intertwines the contractions. We conclude that $\Theta(A) = K[s]$. Going through the definition of the G-action, we find that $\gamma \in K^*$ acts on the affine line by multiplication with γ^{-1}.

Conversely, take $R = K[s] = K[\mathbb{A}^1]$ with K^* acting on $\mathbb{A}^1$ by (inverse) scalar multiplication. Which regular functions $f : \mathbb{A}^1 \to T = K[x, e]$ are equivariant? Write $f(p) = \sum_{i,j} c_{ij}(p) x^i e^j$, so that

$$(f\gamma)(p) = \sum_{i,j} c_{ij}(\gamma^{-1}p)\gamma^{j-i} x^i e^j$$

for the right-hand side to be equal to $f(p)$ for all p, we need that $c_{ij}(\gamma^{-1}p) = \gamma^{i-j} c_{ij}(p)$ for all p, so that c_{ij} is a scalar multiple of s^{i-j}. But for this to be a regular function on $\mathbb{A}^1$, we need $i \geq j$. Hence f is of the form

$$f = \sum_{i \geq j} s^{i-j} x^i e^j.$$

Finally, Derksen needs ideals of tensalgebras: an ideal in a tensalgebra A is by definition the kernel of a tensalgebra homomorphism from A, or, equivalently, a bigraded ordinary ideal which is closed under contractions. They are in one-to-one correspondence with G-stable ideals in $\Theta(A) =: R$, in the following sense: $A \cong (R \otimes T)^G$, and any G-stable ideal J in R gives rise to an epimorphism $A \to (R/J \otimes T)^G$ of tensalgebras, hence to an ideal in A.

Proof of the main theorem. Let A be a sub-tensalgebra of T, and suppose that A is non-degenerate relative to $\langle ., . \rangle$. This latter condition implies that A has no tensalgebra ideals other than 0 and A: if J is is a non-zero ideal, and $v \in J_q^p$ is non-zero, then exists a $w \in A_p^q$ with $\langle v, w \rangle \neq 0$. Now this latter element is a repeated contraction of the element $vw \in J$ to an element of $A_0^0 = K$, hence $J = A$.

The inclusion $A \subseteq T$ gives $\Theta(A) \subseteq R := \Theta(T) = K[G]$. So, in other words, $A \cong (R \otimes T)^G$, where R is some G-stable subalgebra of $K[G]$. Since A is simple, R has no non-zero G-stable ideals. On the other hand, the set of $f \in R$ for which R_f is a finitely generated algebra is an ideal in R, and G-stable by symmetry. Moreover, one can shows that it is non-zero, so that it must be all of R. Hence R is a finitely generated G-stable subalgebra of $K[G]$, and therefore $X := \operatorname{Spec} R$ is an affine G-variety equipped with a dominant G-equivariant map $G \to X$. Since R has no non-trivial G-stable ideals, this map is in fact surjective, so $X = G/H$ for some H, where H is a reductive closed subgroup of G since X is affine.

We conclude that $\Theta(A) \cong K[G/H] \cong \Theta(T^H)$, from which one readily concludes that $A = T^H$. $\square$