An introduction to tropical geometry

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Part I: what you must know

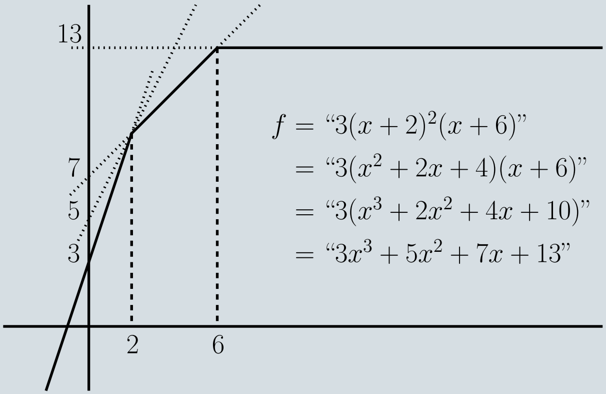
Tropical numbers

$\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ *the tropical semi-ring*
$a \oplus b := \min\{a, b\}$
$a \odot b := a + b$

$\infty \oplus b = b$
$0 \odot b = b$
$\infty \odot b = \infty$
$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

Univariate tropical polynomials

$f = \bigoplus_{i=0}^{d} c_i \odot x^{\odot i}$
shorter: $f =$ "$\sum_{i=0}^{d} c_i x^i$"
can be tropically added and multiplied
f determines function $\overline{\mathbb{R}} \to \overline{\mathbb{R}},\ x \mapsto \min_{i=0}^{d}(c_i + ix)$



Theorem
$g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ piecewise linear, concave, integral slopes
$\rightsquigarrow g$ determined by unique "$c_d \prod_{i=0}^{d}(x + x_i)$" ($x_i$ are **roots** of g)

Relation to ordinary polynomials

K a field
$v : K \longrightarrow \overline{\mathbb{R}}$ a non-Archimedean valuation:

- $v^{-1}(\infty) = \{0\}$
- $v(ab) = v(a) + v(b) (= v(a) \odot v(b))$
- $v(a+b) \geq \min\{v(a), v(b)\} (= v(a) \oplus v(b))$

Examples
$K = \mathbb{Q}$, $v(a) = v_p(a) =$ number of factors p in a
$K = \mathbb{C}((t))$ Laurent series, $v(a_i t^i + \text{ higher order terms}) = i$

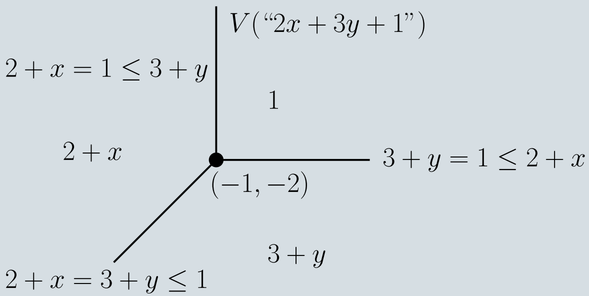
Tropicalisation map
$\mathrm{Trop} : K[x] \longrightarrow \overline{\mathbb{R}}[x],\ \sum_{i=0}^{d} a_i x^i \mapsto \text{``}\sum_{i=0}^{d} v(a_i) x^i\text{''}$

Fundamental fact (Gauss)
$\mathrm{Trop}(fg) = \mathrm{Trop}(f) \odot \mathrm{Trop}(g)$
$\rightsquigarrow \{\text{roots of } \mathrm{Trop}(f)\} = v(\{\text{roots of } f\})$

Multivariate tropical polynomials

$x = (x_1, \ldots, x_n)$
$f = \text{“}\sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha\text{”}$
(only finitely many $c_\alpha \neq \infty$)
determines concave, piecewise linear function $\overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ with integral slopes

Tropical hypersurface
$V(f) := \{x \in \overline{\mathbb{R}}^n \mid f$ not linear at $x\}$
$= \{x \in \overline{\mathbb{R}}^n \mid \min_\alpha(c_\alpha + x \cdot \alpha)$ is attained at least twice$\}$.

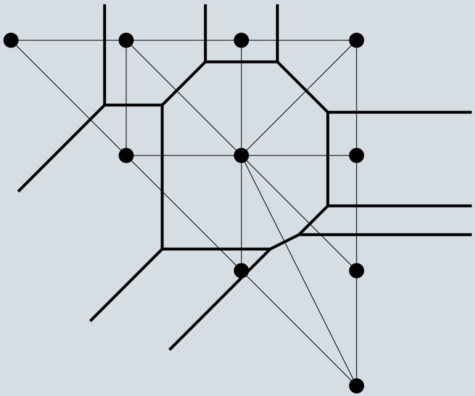


$\rightsquigarrow$ tropical analogues of Desargues, Pappus, etc.

Tropical curves in the plane

$f =$ "$\sum_{i+j\leq d} c_{ij}x^iy^j$"; assume $c_{00}, c_{d0}, c_{0d} \neq \infty$
$C =$ convex hull in $\mathbb{R}^3$ of $\{(i,j,t) \mid t \geq c_{ij}, i,j \in \mathbb{N}, i+j \leq d\}$
edges of C project to line segments in Δ_d spanned by $(0,0), (d,0), (0,d)$
perpendicular to segments of $V(f)$

Fact
C has edge whose projection connects (i,j) and (i',j')
$\Leftrightarrow V(f)$ has segment where minimum is attained exactly in
$c_{ij} + ix + jy$ and in $c_{i',j'} + i'x + j'y$

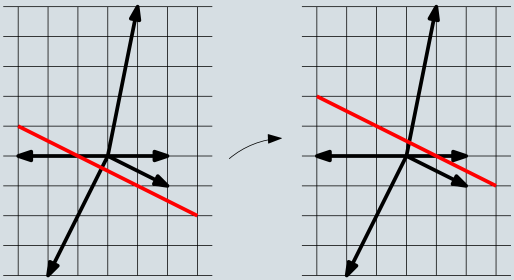


Curves in the plane, continued

Segments of $V(f)$ have *weight* $\gcd(i - i', j - j')$
and are *balanced* around each vertex

Theorem (Richter-Gebert, Sturmfels, Theobald 2003)
X balanced, weighted, piecewise linear curve in $\mathbb{R}^2$
with integral slopes and unbounded segments only in
directions $(1, 0)$, $(0, 1)$, $(-1, -1)$, d each
$\rightsquigarrow X = V(f)$ for suitable f

Theorem (RG-S-T): tropical Bézout
The *stable intersection multiplicity* of tropical curves of degrees d and e is $d \cdot e$.



$$\left|\det \begin{bmatrix} +2 & -2 \\ -1 & -4 \end{bmatrix}\right| + \left|\det \begin{bmatrix} +2 & -3 \\ -1 & +0 \end{bmatrix}\right| = \left|\det \begin{bmatrix} +2 & +2 \\ -1 & +0 \end{bmatrix}\right| + \left|\det \begin{bmatrix} +2 & +1 \\ -1 & +5 \end{bmatrix}\right|$$

$\rightsquigarrow$ tropical analogues of Riemann-Roch, Torelli, Brill-Noether theory etc.

Tropical varieties

Recall: (K, v) valued field
$x = (x_1, \ldots, x_n)$
$\mathrm{Trop} : K[x] \to \overline{\mathbb{R}}[x]$

assume K algebraically closed
$X \subseteq K^n$ algebraic variety
$I \subseteq K[x_1, \ldots, x_n]$ ideal of X

Tropicalisation of X
$\mathrm{Trop}(X) := \bigcap_{f \in I} V(\mathrm{Trop}(f))$

Fundamental theorem of tropical geometry
$v(X) \subseteq \overline{\mathbb{R}}^n$ equals $\mathrm{Trop}(X) \cap v(K)^n$

(Einsiedler-Kapranov-Lind, Speyer-Sturmfels, D, Jensen-Markwig-Markwig, Payne, ...)

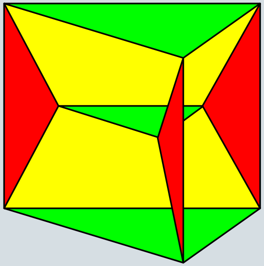
($\subseteq$ easy, $\supseteq$ harder)

Singular matrices

$X \subseteq K^{3\times 3}$ defined by
$x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{13}x_{22}x_{31} - x_{12}x_{21}x_{33} = 0.$

$\mathrm{Trop}(X) \cap \mathbb{R}^{3\times 3}$ consists of $x \in \mathbb{R}^{3\times 3}$ where
$\min\{x_{11} + x_{22} + x_{33}, \ldots, x_{12} + x_{21} + x_{33}\}$ attained $\geq$ twice.

- one 8-dimensional cone for each pair of permutations
- all cones stable under adding $y_i + z_j$ to position (i, j)
- modulo this, 3-dimensional facets in 4-space
- intersecting with 3-sphere in 4 space gives



Tropical varieties, continued

Theorem (Bieri-Groves)
$X \subseteq K^n$ pure of dimension d
$\Rightarrow \mathrm{Trop}(X)$ pure polyhedral complex of dimension d

Grassmannian of 2-spaces
$X \subseteq K^{\binom{m}{2}}$
defined by relations among
$x_{ij} = y_i z_j - y_j z_i$
$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$

Theorem (Speyer-Sturmfels)
Tropicalisations of these quadratic three-term relations cut out $\mathrm{Trop}(X)$
$\rightsquigarrow$ (phylogenetic) trees

Part II: three tropical challenges

Challenge 1: reparameterisations

Lemma

$\phi : K^m \to K^n$ polynomial map, $X = \overline{\operatorname{im}\phi}$
$\Rightarrow \operatorname{Trop}(\phi) : \overline{\mathbb{R}}^m \to \operatorname{Trop}(X) \subseteq \overline{\mathbb{R}}^n$
(typically strict containment)

Question

$\exists$? finitely many *reparameterisations*
$\alpha : K^p \to K^m$ such that
$\operatorname{Trop}(X) = \bigcup_\alpha \operatorname{im}\operatorname{Trop}(\phi \circ \alpha)$

Challenge 2: secant varieties

Secant varieties of $(\mathbb{P}^1)^n$:
n natural number $\neq 4$
$C = \{0, 1\}^n$ hypercube
$k = \lfloor \frac{2^n}{n+1} \rfloor$
By repeated cutting with hyperplanes, can you partition C into k affine bases of $\mathbb{R}^n$ and 1 affinely independent set?

(Lots of variations for Segre-Veronese varieties!)

Challenge 3: B-N theory for graphs

Requirements

finite, undirected graph Γ

$d \geq 0$ chips

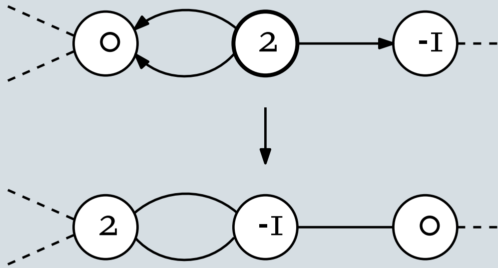
natural number r

Rules

B puts d chips on Γ

N demands $r_v \geq 0$ chips at v with $\sum_v r_v = r$

B wins iff he can *fire* to meet N's demand



Brill-Noether theorems for graphs

$g := e(\Gamma) - v(\Gamma) + 1$ *genus* of Γ
$\rho := g - (r+1)(g-d+r)$

Conjecture (Baker)
1. $\rho \geq 0 \Rightarrow$ B has a winning starting position.
2. $\rho < 0 \Rightarrow$ B may not have one, depending on Γ.
($\forall g \; \exists \Gamma \; \forall d, r : \rho < 0 \Rightarrow$ Brill loses.)

Theorem (Baker)
1. is true if B may put chips at rational points of edges.
(*uses* sophisticated algebraic geometry)

Theorem (Cools-D-Payne-Robeva)
2. is true.
(*implies* sophisticated algebraic geometry)