AN INTRODUCTION TO TROPICAL GEOMETRY

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1. Tropical numbers and valuations

Tropical numbers.

- $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ *tropical numbers*, equipped with two operations: *tropical addition* "$a + b$" $= \min\{a, b\}$ and *tropical multiplication* "ab" $= a + b$
- commutative, associative
- "$a(b + c)$" = "$ab + ac$" for all $a, b \in \mathbb{R}_\infty$
- neutral elements: ∞ for tropical addition, 0 for tropical multiplication; note: *we do NOT write* "1" $= 0$ *or* "0" $= \infty$.
- "$(a + b)^n$" = "$a^n + b^n$" for all $a, b \in \mathbb{R}_\infty$
- note that "$x + (-0.5)$" $\neq$ "$x - 0.5$"—in fact, the latter expression has no meaning. Note that we do not write "1/2"—if anything, that would be the real number -1..
- tropical matrix multiplication: replace $+$ by min and $\cdot$ by $+$.

Exercise 1.1. Define the term *(tropical) eigenvector* and the corresponding *eigenvalue* for tropical matrices $A \in \mathbb{R}_\infty^{n\times n}$. Show that an eigenspace is closed under (componentwise) tropical addition and under tropical scalar multiplication. Draw the eigenspaces of each of the following matrices in $\mathbb{R}_\infty^2$:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix},\ \text{and } C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Tropical polynomials.

- *tropical polynomials* "$\sum_{i=0}^d a_i x^i$", can be tropically added and multiplied
- defines a piecewise linear function, concave function $\mathbb{R}_\infty \to \mathbb{R}_\infty$ with integral slopes
- cannot be reconstructed from that function

Theorem 1.2. *For every piecewise linear, concave function $f : \mathbb{R}_\infty \to \mathbb{R}_\infty$ with integral slopes that is not identically ∞, there exist unique $d \in \mathbb{N}$ and unique $x_1, \ldots, x_d \in \mathbb{R}_\infty$ (up to permutation) and unique $c \in \mathbb{R}$ such that $f(x) =$ "$c(x + x_1)\cdots(x + x_d)$" for all $x \in \mathbb{R}_\infty$.*

The numbers x_i are called the *roots* of f. The finite roots are found as follows: take the positions $x \in \mathbb{R}$ where f is non-linear. Then the slope of f decreases by a positive integer m. Then x is a root of multiplicity m. Similarly, the multiplicity of ∞ is the (constant) slope of f at $x \gg 0$. Finally, the coefficient c is determined by specialising at a suitable value.

Fields with a valuation.

- K a field
- *valuation* $v : K \to \mathbb{R}_\infty$ satisfying $v(ab) = v(a) + v(b)$ (= "$v(a)v(b)$") and $v(a + b) \geq \min\{v(a), v(b)\}$ (= "$v(a) + v(b)$") and $v^{-1}(\infty) = \{0\}$
- easy but important facts: $v(1) = 0$ and $v(-a) = v(a)$ and $v(a+b) = v(a)$ if $v(a) < v(b)$. (For the latter, write $v(a) = v((a+b)-b) \geq \min\{v(a+b), v(b)\}$, so $v(a) \geq v(a + b)$ since $v(a) < v(b)$.)
- examples: $\mathbb{Q}$ with p-adic valuation, $\mathbb{C}(t)$ (rational functions) or $\mathbb{C}((t))$ (Laurent series) with t-adic valuations
- fundamental fact: valuation can be extended to any field extension; discuss completion and algebraic closure

Definition 1.3. The *tropicalisation* of $f = \sum_{i=0}^{d} c_i x^i \in K[x]$ is $\mathrm{Trop}(f) :=$ "$\sum_{i=0}^{d} v(c_i) x^i$".

Lemma 1.4 (Gauss's lemma). *For $f, g \in K[x]$ we have* $\mathrm{Trop}(fg)(a) =$ "$\mathrm{Trop}(f)(a)\,\mathrm{Trop}(g)(a)$" *for all $a \in \mathbb{R}_\infty$.*

Exercise 1.5. The preceding lemma concerns the *functions* defined by $\mathrm{Trop}(fg)$, $\mathrm{Trop}(f)$, and $\mathrm{Trop}(g)$. Find a concrete example of polynomials f, g, say over the field $\mathbb{C}(t)$, where $\mathrm{Trop}(fg)$ is *not* equal to "$\mathrm{Trop}(f)\,\mathrm{Trop}(g)$" *as polynomials.*

Theorem 1.6 (Newton? Puiseux?). *For $f \in K[x]$ with all roots in K we have*

$$v(\{\textit{roots of } f\}) = \{\textit{roots of } \mathrm{Trop}(f)\}.$$

Exercise 1.7. Using the previous theorem and the 2-adic valuation on $\mathbb{Q}$, give a "tropical proof" of the non-rationality of $\sqrt{2}$.

Exercise 1.8. Eisenstein's criterion says that if a polynomial $f = c_d x^d + \ldots + c_0 \in \mathbb{Z}[x]$ satisfies $p \not| c_d$, $p | c_{d-1}, \ldots, c_0$, and $p^2 \not| c_0$, then f is irreducible over the rational numbers. Take v equal to the p-adic valuation on $\mathbb{Q}$. Prove that, in this case, $\mathrm{Trop}(f)$ has a d-fold root $1/d$, prove that the function defined by $\mathrm{Trop}(f)$ is "irreducible", and argue that this implies Eisenstein's criterion.

2. Tropicalising varieties

Tropical hypersurfaces.

Definition 2.1. A tropical multivariate polynomial $f =$ "$\sum_{\alpha\in\mathbb{N}} c_\alpha x^\alpha$" defines a piecewise linear map $f : \mathbb{R}^n_\infty \to \mathbb{R}_\infty$. Define

$$\mathcal{T}(f) := \{x \in \mathbb{R}^n_\infty \mid f \text{ is infinite or else not linear at } x\},$$

the *tropical hypersurface* defined by f (or *corner locus* of f).

Exercise 2.2. Prove that $\mathcal{T}(f)$ consists of all points where either f takes the value ∞ or else there are at least two distinct α, β with $c_\alpha + \alpha \cdot x = c_\beta + \beta \cdot x$, i.e., where *the minimum is attained at least twice.*

Example 2.3. $f =$ "$x_{11}x_{22} + x_{12}x_{21} + 0$" $= \mathrm{Trop}(\det -1)$. Note: $\mathcal{T}(f) \subseteq \mathbb{R}^{2\times 2}_\infty$ is closed under tropical matrix multiplication. "Tropical SL_2."

Tropicalising varieties. Recall: K field with valuation v (perhaps trivial). We assume K *algebraically closed.*

Definition 2.4. For an algebraic variety $X \subseteq K^n$ defined by ideal $I \subseteq K[x_1, \ldots, x_n]$ define

$$\mathrm{Trop}(X) := \bigcap_{f\in I} \mathcal{T}(\mathrm{Trop}(f)),$$

the *tropicalisation* of X.

Remark 2.5. $\mathrm{Trop}(X)$ doesn't depend on the ideal I chosen to define X: take $J = \sqrt{I}$. Since J contains I, the tropicalisation of X using J is contained in that using I. But since every $f \in J$ has a power f^k in I (Hilbert's Nullstellensatz), and since $\mathcal{T}(\mathrm{Trop}(f)) = \mathcal{T}(\mathrm{Trop}(f^k))$, we also have the opposite inclusion.

Exercise 2.6. Prove that if I is generated by a single element f, then $\mathrm{Trop}(X)$ is just $\mathcal{T}(\mathrm{Trop}(f))$. (Use Gauss's lemma.)

Example 2.7. $X = \mathrm{O}_2(\mathbb{C}) = \{x \in \mathbb{C}^{2\times 2} \mid x^T x = I\}$, say over $\mathbb{C}$ with trivial valuation. So X is defined by the vanishing of $f_1 := a^2 + c^2 - 1$ and $f_2 := b^2 + d^2 - 1$ and $f_3 := ab + cd$. Tropicalising f_3 yields that $a + b = c + d$ holds on $\mathrm{Trop}(X)$. Tropicalising f_1 yields that $\min\{2a, 2c, 0\}$ is attained (at least) twice, which gives $\min\{a, c, 0\}$ attained twice. Similarly, f_2 gives $\min\{b, d, 0\}$ attained twice. All conditions are closed under multiplication with a common positive scalar, so we distinguish three cases:

(1) $a = 0$. Then $c \geq 0$ by f_1. If $c > 0$ then $b = d + (c - a) = b + c > d$ by f_3 and hence $d = 0$ by f_2 and hence $b = c$. Thus we have the cone

$$C_1 := \left\{ \begin{bmatrix} 0 & c \geq 0 \\ c & 0 \end{bmatrix} \right\},$$

(which includes the case where $c = \infty$).

If $c = 0$ then $b = d$ by f_3 and $b = d \leq 0$ by f_2. Thus we have the cone

$$C_2 := \left\{ \begin{bmatrix} 0 & b \leq 0 \\ 0 & b \end{bmatrix} \right\}.$$

(2) $a > 0$. Then $c = 0$ by f_1 and $a + b = d$ by f_3, so $d > b$ so $b = 0$ by f_2, and $d = a$. This gives the cone $C_3 = C_1^T$.

(3) $a < 0$. Then $c = a$ by f_1 and $b = d$ by f_3, so $b = d \leq 0$ by f_2. This gives a two-dimensional cone

$$C_4 := \left\{ \begin{bmatrix} a \leq 0 & b \leq 0 \\ a & b \end{bmatrix} \right\}.$$

The union $C := C_1 \cup C_2 \cup C_3 \cup C_4$ is not stable under transposition, while O_2 is. Using the "transposed equations" $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, $ac + bd = 0$ for O_2 we find $C^T := C_1^T \cup C_2^T \cup C_3^T \cup C_4^T$. So certainly $\mathrm{Trop}(O_2)$ is contained in the intersection of C and C^T. This intersection equals $C_1 \cup C_3 \cup C_5$, where

$$C_5 := \left\{ \begin{bmatrix} a \leq 0 & a \\ a & a \end{bmatrix} \right\}$$

is contained in C_4. Note that C_1, C_3, C_5 all have dimension 1, and that any two of them meet in the all-zero matrix. Is their union equal to $\mathrm{Trop}(O_2)$?

(1) $X \subseteq K^n$ variety as before
(2) L any valued extension of K, and $p = (p_1, \ldots, p_n) \in X(L)$. Then $v(p) \in \mathrm{Trop}(X)$: Write $x_i = v(p_i)$ and take $f = \sum_\alpha c_\alpha x^\alpha \in I$. Then $v(c_\alpha) + \alpha \cdot x$ is the valuation of the α-term in f. If the minimum of these valuations were finite and attained only once, then $v(f(p))$ would equal that minimum. But this contradicts $v(f(p)) = v(0) = \infty$. Hence the minimum is either infinite or attained at least twice.
(3) This gives a method of *certifying* that a point lies in $\mathrm{Trop}(X)$, by given a *lift* in $X(L)$ for suitable L.

Example 2.8. In the O_2-example take a point in C_3 with $a > 0$ and construct $L = \mathbb{C}((t))$ with the scaled p-adic valuation where where $v(t) = a$. Then the valuation of the orthogonal matrix

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

equals the prescribed matrix. This shows that $C_3 \subseteq \mathrm{Trop}(O_2)$.

Exercise 2.9. Find lifts of arbitrary points in C_1 and C_5 in suitable valued extensions of $\mathbb{C}$, showing that $C_1 \cup C_3 \cup C_5$ is really the tropicalisation of O_2.

Exercise 2.10. Prove that $\mathrm{Trop}(O_2)$ is closed under tropical matrix multiplication.

Tropical basis theorem.

Theorem 2.11. *There exist finitely many $f_1, \ldots, f_k \in I$ such that* $\mathrm{Trop}(X)$ *is the intersection of all* $\mathcal{T}(\mathrm{Trop}(f_i))$.

Such a tuple (sometimes required, in addition, to generate the ideal I) is called a *tropical basis* of the ideal. These can be *computed* (but not very efficiently).

Fundamental theorem.

Theorem 2.12. *$X \subseteq K^n$ algebraic variety, K algebraically closed. Then for every point a in* $\mathrm{Trop}(X)$ *with coordinates in $v(K)$ there exists a point $x = (x_1, \ldots, x_n) \in X(K)$ with $v(x) = a$.*

Bieri-Groves's theorem.

- *polyhedron* in $\mathbb{R}^n$: intersection of a finite number of closed half-spaces.
- *dimension* is the dimension of the smallest affine subspace containing it.
- polyhedron in $\mathbb{R}^n_\infty$: topological closure of a polyhedron in some $\mathbb{R}^m \times \{\infty\}^{n-m}$ (up to permutation).

Theorem 2.13. $\mathrm{Trop}(X)$ *can be written as a finite union of polyhedra. If, moreover,* X *is irreducible of dimension* d*, then the polyhedra can all be chosen of dimension* d*.*

Exercise 2.14. Show that for varieties $X, Y \subseteq K^n$ one has $\mathrm{Trop}(X) \cup \mathrm{Trop}(Y) = \mathrm{Trop}(X \cup Y)$. Give one proof using the definition of Trop and one proof using the fundamental theorem.

The fundamental theorem for hypersurfaces. We follow Payne's proof [5].

- K valued field
- $R := \{c \in K \mid v(c) \geq 0\}$ *valuation ring*
- $M := \{c \in K \mid v(c) > 0\}$ *maximal ideal*
- $k := R/M$ *residue field* (algebraically closed if K is).
- homomorphisms $c \mapsto \bar{c}$ from $R \to R/M = k$, extends to polynomials.
- Now $X \subseteq K^n$ given by a single polynomial f.
- Given $a \in \mathrm{Trop}(X) = \mathcal{T}(f)$ with valuation $v(a) \in v(K)^n$, want to lift a to a point $p \in X(K)$ with valuation a.
- Easy reduction to case where $a = (0, \ldots, 0)$, which we assume now, so after dividing f by the coefficient with smallest valuation have $f \in R[x]$ with at least two coefficients not in M.
- Write $\bar{f} \in k[x]$ for image of f modulo M; this has at least two terms. Hence there is a variable x_i that appears in $\bar{f}$ with at least two distinct exponents; wlog $x_i = x_n$.
- Write $f = f_0 + f_1 x_n \ldots + f_d x_n^d + \ldots f_e x_n^e + \ldots + f_r x_n^r$ with $f_0, \ldots, f_r \in K[x_1, \ldots, x_{n-1}]$ and $d < e$ and $\bar{f}_d, \bar{f}_e \neq 0$.
- Choose a point $\bar{q}$ in $(k^*)^{n-1}$ where $\bar{f}_d, \bar{f}_e$ are non-zero, and lift the point to $q = (q_1, \ldots, q_{n-1}) \in R \setminus M$.
- Hence $f_d(q_1, \ldots, q_{n-1})$ and $f_e(q_1, \ldots, q_{n-1})$ have valuation zero.
- Set $g(y) := f(q_1, \ldots, q_{n-1}, y) \in R[y]$, a univariate polynomial. It satisfies:
  1. $\mathrm{Trop}(g)(0) = 0$.
  2. $\bar{g} = \bar{f}(\bar{q}, y)$ has at least two non-zero terms, and hence a non-zero root in k^*; lift this root to $\tilde{q_n} \in R \setminus M$.
  3. $v(g(\tilde{q_n})) > 0$ while $v(\tilde{q_n}) = 0$.
- Hence $\mathrm{Trop}(g)$ has a tropical root at zero.
- But that means that g has a root q_n with $v(q_n) = 0$ (the corollary to Gauss's lemma).
- $(q_1, \ldots, q_n)$ is the required root of f with valuation 0.

Note that $q_1, \ldots, q_{n-1}$ were more or less "freely" chosen (subject to the non-vanishing of $\bar{f}_d, \bar{f}_e$). This can be used to prove that the image under projection on the first $(n-1)$ coordinates of the fibre $v^{-1}(0) \cap X = (R \setminus M)^n \cap X$ is Zariski-dense in K^{n-1}. As a consequence, if X is irreducible and not contained in any coordinate hyperplane, then $v^{-1}(a) \cap X$ is Zariski-dense in X for every $a \in \mathrm{Trop}(X) \cap \mathbb{R}^n$. We will use this later on.

Well-behaved under monomial maps. The following proposition shows that tropical varieties behave "linearly" under monomial maps.

Proposition 2.15. *$X \subseteq K^n$ algebraic variety, and $\pi : K^n \to K^m$ a monomial map, i.e., $\pi(x) = (x^{\alpha_1}, \ldots, x^{\alpha_m})$ for certain $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^n$. Let A be the $m \times n$-matrix with rows the α_i, and let Y be the closure of $\pi(X)$. Then $\mathrm{Trop}(Y) \supseteq A \cdot \mathrm{Trop}(X)$, where $\cdot$ is just matrix-column multiplication.*

Actually, equality holds, but this will follow only later. The proposition can be proved using the fundamental theorem, but we will want to use it to *prove* the fundamental theorem, so we proceed from first principles.

- I an ideal defining X
- J the ideal of all polynomials in $K[y_1, \ldots, y_m]$ that are mapped into I when each y_i is replaced by x^{α_i}.
- Then J is an ideal defining Y.
- $u \in \mathbb{R}^n_\infty$ be a point in $\mathrm{Trop}(X)$
- $f = \sum_\beta c_\beta y^\beta \in J$. We claim that $\mathrm{Trop}(f)$ is infinite or not linear at Au; this will then prove that Au lies in $\mathrm{Trop}(Y)$.
- Replacing y_i by x^{α_i} in f yields the polynomial $\sum_\beta c_\beta x^{\beta \cdot A}$ where β is considered a row vector.
- Tropicalising gives the tropical polynomial $\min_\beta(v(c_\beta) + \beta \cdot A \cdot x)$, and by assumption, at $x = u$, this minimum is either infinite or else attained at least twice.
- This is equivalent to the statement that $\mathrm{Trop}(f)$ is infinite or not linear at $A \cdot x$.

This proves that $A \cdot \mathrm{Trop}(X) \subseteq \mathrm{Trop}(Y)$.

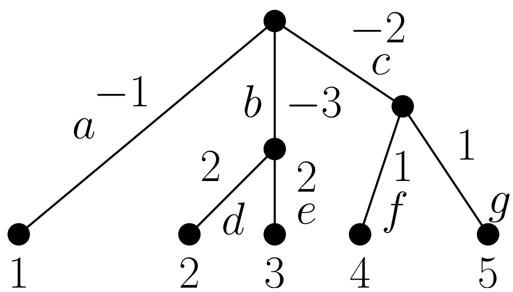
Exercise 2.16. Find a 2×2-matrix over the field $\mathbb{C}(t)$ with determinant 0 and entrywise valuation $\begin{bmatrix} 1 & 5 \\ -2 & 2 \end{bmatrix}$.

Tropical Grassmannian and space of trees. For this stuff see [6].

- Plücker map

$$K^n \to K^{\binom{n}{2}}, x = \begin{bmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{bmatrix} \mapsto (z_{ij} := x_i y_j - x_j y_i)_{i<j}$$

- Image defined by ideal I generated by the polynomials (for $i < j < k < l$): $z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk}$ This is the (affine cone over the) *Grassmannian.*
- Theorem (Speyer-Sturmfels): these form a *tropical basis*; *the Tropical Grassmannian.*
- How to make tuples $(z_{ij}) \in \mathbb{R}_\infty^{\binom{n}{2}}$ such that for each $i < j < k < l$ the minimum of $z_{ij} + z_{kl}$ and $z_{ik} + z_{jl}$ and $z_{il} + z_{jk}$ is attained at least twice? (Have to lift such points to the tropical variety.)
- Answer: trees with n leaves, negative weights on internal edges, and arbitrary weights on leaf edges.
- Theorem (neighbour-joining): for every tuple there is a tree with that tuple as leaf-to-leaf distance matrix.
- Construction: from trees to matrices; first reduce to distance-balanced case.



$$x_1 = at^{-2}$$
$$x_2 = bt^{-2} + dt^4$$
$$x_3 = bt^{-2} + et^4$$
$$x_4 = ct^{-2} + ft^2$$
$$x_5 = ct^{-2} + gt^2$$

3. Proofs of some fundamental results

$X \subseteq K^n$ algebraic variety defined by an ideal I.

Theorem 3.1 (Existence of finite tropical bases)**.** *There exist finitely many $f_1, \ldots, f_k \in I$ such that* $\mathrm{Trop}(X)$ *is the intersection of all* $\mathcal{T}(\mathrm{Trop}(f_i))$*.*

Theorem 3.2 (Fundamental Theorem)**.** *$X \subseteq K^n$ algebraic variety, K algebraically closed. Then for every point a in* $\mathrm{Trop}(X)$ *with coordinates in $v(K)$ there exists a point $x = (x_1, \ldots, x_n) \in X(K)$ with $v(x) = a$.*

Done for X a hypersurface, in fact set of such points is then dense in X.

Theorem 3.3 (Bieri-Groves)**.** $\mathrm{Trop}(X)$ *can be written as a finite union of polyhedra. If, moreover, X is irreducible of dimension d, then the polyhedra can all be chosen of dimension d.*

Proof of Bieri-Groves's Theorem. We follow Bieri and Groves [1].

- X irreducible algebraic variety of dimension d in K^n
- to show: $\mathrm{Trop}(X)$ is a union of d-dimensional polyhedra.
- if X hypersurface with equation f, then done ($\mathrm{Trop}(X)$ union of polyhedra dual to the induced subdivision of the Newton polytope of f).

Lemma 3.4. *If $A \in \mathbb{N}^{(d+1)\times n}$ such that the induced monomial map $\pi_A : K^n \to K^{d+1}$ maps X onto a hypersurface Y, then* $\mathrm{Trop}(Y) = A\,\mathrm{Trop}(X)$*.*

Proof. We know $\mathrm{Trop}(Y) \supseteq A\,\mathrm{Trop}(X)$. For the converse, let a be a point in $\mathrm{Trop}(Y)$. Then the set $\{y \in Y \mid v(y) = a\}$ is non-empty (hypersurface case of fundamental theorem) and in fact Zariski-dense in Y. Hence it intersects the constructable set $\pi(X)$ (which contains an open dense subset of Y by basic algebraic geometry). Hence there is $x \in X$ be such that $v(\pi(x)) = a$. This translates into $A \cdot v(x) = a$, while of course $v(x) \in \mathrm{Trop}(X)$. □

Here's the key idea, which we will not prove but which is at least plausible.

Lemma 3.5 (Regular projection lemma)**.** *If a set S in $\mathbb{R}^n_\infty$ has the property that for "generic" matrices $A \in \mathbb{N}^{(d+1)\times n}$ the image $A \cdot S$ of S is a finite union of d-dimensional polyhedra, then so is S itself. Moreover, one can find finitely many matrices $A_1, \ldots, A_k$ such that*

$$S = \bigcap_{i=1}^{k} A_i^{-1}(A_i(S)).$$

Remark 3.6. In fact, one can find $k = n - d + 1$ such matrices, which proves the existence of a tropical basis of that cardinality (at least, if one does not require that a tropical basis generate the ideal).

- "π_A maps X onto a hypersurface" turns out to be sufficiently generic
- for each such A, $A\,\mathrm{Trop}(X)$ is a finite union of d-dimensional polyhedra by the hypersurface case of Bieri-Groves
- applying the regular projection lemma yields that so is $\mathrm{Trop}(X)$.
- in fact, it yields more: take $A_1, \ldots, A_k$ as in the lemma, and $f_1(y), \ldots, f_k(y)$ the equations of the corresponding hypersurfaces in K^{d+1}, and let $g_1, \ldots, g_k$

be their pull-backs to $K[x_1, \dots, x_n]$ under $\pi_{A_1}, \dots, \pi_{A_k}$; these are elements of the ideal of X. Then the lemma says

$$\mathrm{Trop}(X) = \bigcap_{i=1}^{k} \mathcal{T}(\mathrm{Trop}(g_i)),$$

so $g_1, \dots, g_k$ forms a tropical basis.

This latter proof of existence of a finite tropical basis first appeared in print in [4], while the first proof appeared in [2].

Proof of Fundamental Theorem. (For an affinoid proof see [3].)

- Fix $u \in \mathrm{Trop}(X) \cap v(K)^n$. Want to show that there exists an $x \in X(K)$ with $v(x) = u$.
- Reduce to case where X is irreducible
- Reduce to case where $u \in RR^n$.

Lemma 3.7. *Fix a point $u \in \mathrm{Trop}(X) \cap \mathbb{R}^n$. Then for "generic" $A \in \mathbb{N}^{(d+1)\times n}$ such that $A^{-1}(Au) \cap \mathrm{Trop}(X)$ consists only of u.*

The proof will make clear what generic means in this case.

Proof.
- This uses (only) that $\mathrm{Trop}(X)$ is the union of finitely many d-dimensional polyhedra.
- for an affine subspace $V = v_0 + \langle v_1, \dots, v_d \rangle \subseteq \mathbb{R}^n$ the condition that $Av = Au$ translates into $A(u - v_0) \in \langle Av_1, \dots, Av_d \rangle$, i.e., the determinant of the $(d+1) \times (d+1)$-matrix with rows $A(u - v_0), Av_1, \dots, Av_d$ is zero.
- This gives a non-trivial polynomial equation on A that should be *avoided*.
- Only finitely many of these need to be avoided.
- $\mathbb{N}^{(d+1)\times n}$ is Zariski-dense in $\mathbb{R}^{(d+1)\times n}$, so these conditions are avoided by some ("most") $A \in \mathbb{N}^{(d+1)\times n}$.

□

Now for the proof of the fundamental theorem:

- Pick a matrix $A \in \mathbb{N}^{(d+1)\times n}$ as in the lemma, and with the additional property that the corresponding monomial map maps X onto a hypersurface Y in $K^{(d+1)}$. (Intersection of finitely many generic conditions is generic.)
- Pick a point $y \in \pi_A(X)$ with valuation Au (see the proof of Lemma 3.4), and take $x \in X$ with $\pi_A(x) = y$.
- Then $Av(x) = v(y) = Au$ and $v(x) \in \mathrm{Trop}(X)$. Hence by the choice of A, $v(x) = u$.

Actually, this proof shows that points $x \in X$ with $v(x) = u$ are Zariski-dense in X; this is the main result of [5].

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