Tropical Brill-Noether theory

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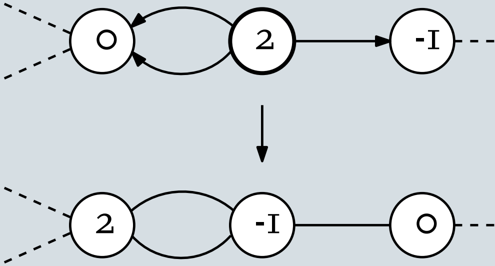
The B(aker)-N(orin) game on graphs

Requirements
finite, connected, undirected graph Γ
$d \geq 0$ chips
natural number r



Rules
B puts d chips on Γ
N demands $\geq r_v \geq 0$ chips at v with $\sum_v r_v = r$
B wins iff he can *fire* to meet N's demand

Brill-Noether theorems for graphs

$g := e(\Gamma) - v(\Gamma) + 1$ *genus* of Γ
$\rho := g - (r+1)(g-d+r)$

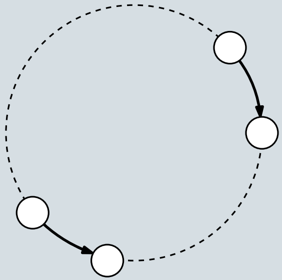
Conjecture (Matthew Baker)
1. $\rho \geq 0 \Rightarrow$ B has a winning starting position.
2. $\rho < 0 \Rightarrow$ B may not have one, depending on Γ.
($\forall g \, \exists \Gamma \, \forall d, r : \rho < 0 \Rightarrow$ Brill loses.)

Theorem (Baker / Caporaso)
1. is true.
(*uses* sophisticated algebraic geometry)

Theorem (Cools-D-Payne-Robeva)
2. is true.
(*implies* sophisticated algebraic geometry)

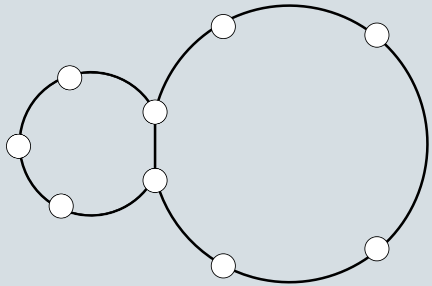
Chip dragging on graphs

Simultaneously moving all chips along edges,
with zero net movement around every cycle.



Lemma
1. Chip dragging is realisable by chip firing.
2. W.l.o.g. B *drags* instead of *firing*.

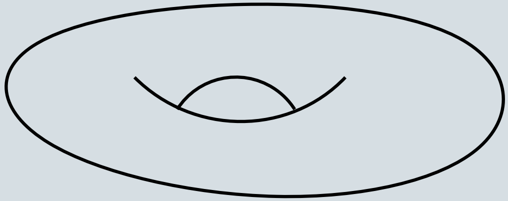
Example 1: Γ a tree
$\rho = g-(r+1)(g-d+r) = -(r+1)(-d+r)$
B wins $\Leftrightarrow \rho \geq 0 \Leftrightarrow d \geq r$



Example 2: a hyperelliptic graph
$d = 2, r = 1$
Who wins?

The B(rill)-N(oether) game on Riemann surface

Requirements
compact Riemann surface X
d chips
natural number r



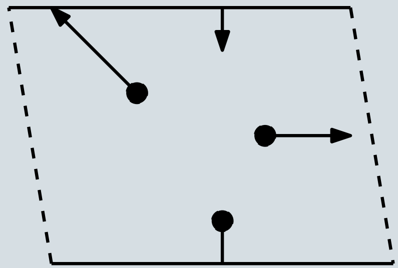
Rules
B puts d chips on X
N demands $\geq r_x \geq 0$ chips at x with $\sum_x r_x = r$
B wins iff he can *drag* to meet N's demand

Chip dragging on Riemann surfaces

Simultaneously moving chips c along paths $\gamma_c : [0,1] \to X$, such that $\sum_c \langle \omega|_{\gamma(t)}, \gamma_c'(t)\rangle = 0$ for all holomorphic 1-forms ω on X.

Lemma
$D = \sum_c [\gamma_c(0)]$ initial position
$E = \sum_c [\gamma_c(1)]$ final position
$\Leftrightarrow E - D$ is *divisor* of meromorphic function on X
drag-equivalence $=$ *linear equivalence*

Example: torus
only one holomorphic 1-form: $\mathrm{d}z$
condition: $\sum_c \gamma_c'(t) = 0$
when does B win?



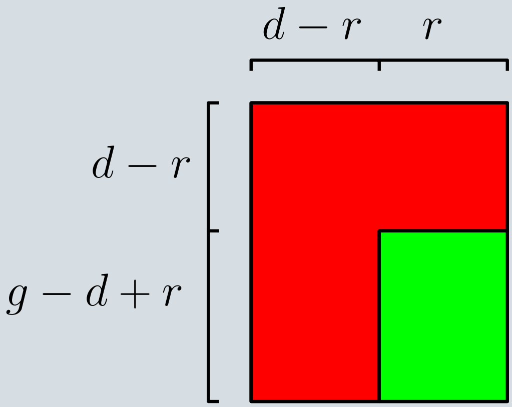
Dimension count

$\omega_1, \ldots, \omega_g$ basis of holomorphic 1-forms
$\mathbf{x} = (x_1, \ldots, x_d) \in X \times \cdots \times X$
$v_i \neq 0$ tangent vector at x_i
$\rightsquigarrow$ matrix $A_{\mathbf{x}} = (\langle \omega_i, v_j \rangle)_{ij} \in \mathbb{C}^{g \times d}$
$(c_1 v_1, \ldots, c_d v_d)$ infinitesimal dragging direction $\Rightarrow A(c_1, \ldots, c_d)^T = 0$

$\mathbf{x}$ winning for $B \Rightarrow$
dragging $\mathbf{x}$ fills $\geq r$-dimensional variety
where $\ker A$ is $\geq r$-dimensional

$\#$ conditions on $g \times d$-matrix to have $\geq r$-dimensional kernel: $r(g-d+r)$

for B to have a winning position, "need"
$d - r(g-d+r) \geq r$
$\Leftrightarrow \rho = g - (r+1)(g-d+r) \geq 0$



Brill-Noether theorems for Riemann surfaces

Theorem (Meis 1960, Kempf 1971, Kleiman-Laksov 1972)
$\rho \geq 0 \Rightarrow$ B has a winning position.

Theorem (Griffiths-Harris 1980)
1. $\rho < 0 \Rightarrow$ B may lose, depending on X.
($\forall g \; \exists X \; \forall d, r : \rho < 0 \Rightarrow$ B loses.)

2. $\rho \geq 0$ and X general
$\Rightarrow \rho = \dim\{$winning positions modulo dragging$\}$

3. $\rho = 0$ and X general
$\Rightarrow \# = \#$ standard tableaux of shape
$(r+1) \times (g-d+r)$ with entries $1, 2, \ldots, g$

Specialisation

Algebro-geometric (Baker, Caporaso)
dual graph of special fibre
applies to arbitrary fields
integral starting positions for B

Complex-analytic (Mikhalkin-Zharkov)
conceptually simpler?
rational starting positions for B
$\{X_t\}_{t \neq 0}$ family of Riemman surfaces
$\rightsquigarrow \Gamma$ for $t \longrightarrow 0$ ("tropical limit")
holomorphic 1-forms on $X_t \rightsquigarrow$ "1-forms" on Γ
chip dragging on $X_t \rightsquigarrow$ chip dragging on Γ

Theorem
D_t winning for B on X_t and $D_t \longrightarrow D$ on Γ for $t \longrightarrow 0$
$\Rightarrow D$ winning on Γ.

Consequences of Specialisation

Meis/Kempf/Kleiman-Laksov
($\rho \geq 0$ implies B wins on Riemann surfaces)
$\Rightarrow$ same statement for Γ.
No combinatorial proof is known!

Cools-D-Payne-Robeva
($\rho < 0 \Rightarrow$ B loses for suitable Γ)
$\Rightarrow$ same for Riemann surfaces (Griffiths-Harris 1 and 2, and probably 3).

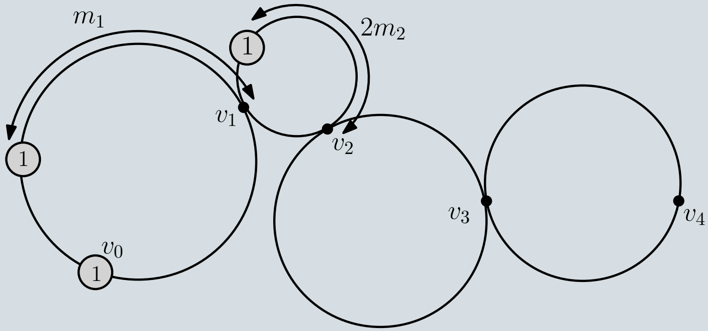
Technical difficulties:
1. Find family $\{X_t\}_t$ with
dual graph Γ (algebro-geometric) or
degenerating to Γ (complex-analytic);
2. show that $t \mapsto D_t$ (winning position on X_t)
can be chosen such that D_t "converges".

Example (Cools-D-Payne-Robeva)

$g = 4, d = 3, r = 1$
$g - d + r = 2$
$r + 1 = 2, \rho = 0$

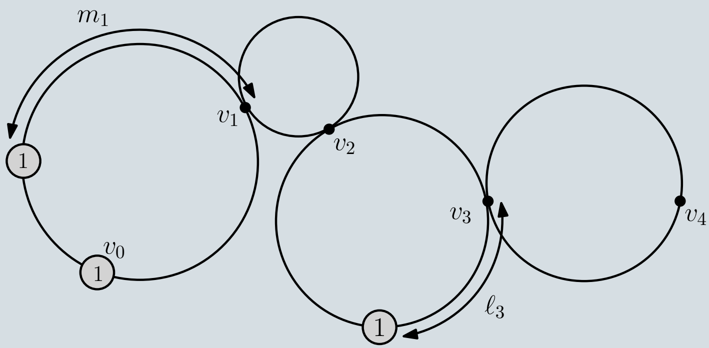
1	3
2	4

$\rightsquigarrow 1, 2, 3, 2, 1$



1	2
3	4

$\rightsquigarrow 1, 2, 1, 2, 1$

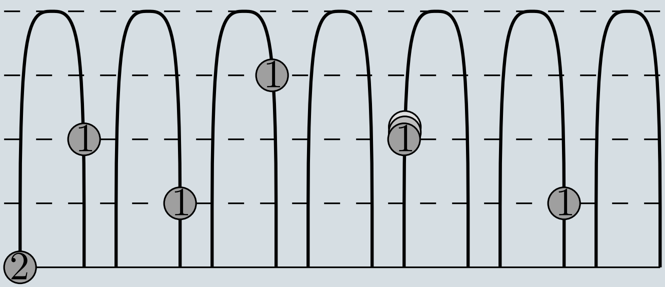


A larger example

$g = 7, d = 7, r = 2$
$\rightsquigarrow g - d + r = 2, r + 1 = 3, \rho = 1$

1	2	4
3	6	7

$\rightsquigarrow (21, 31, 32, 42, 31, 31, 32, 21)$ *lingering lattice path*



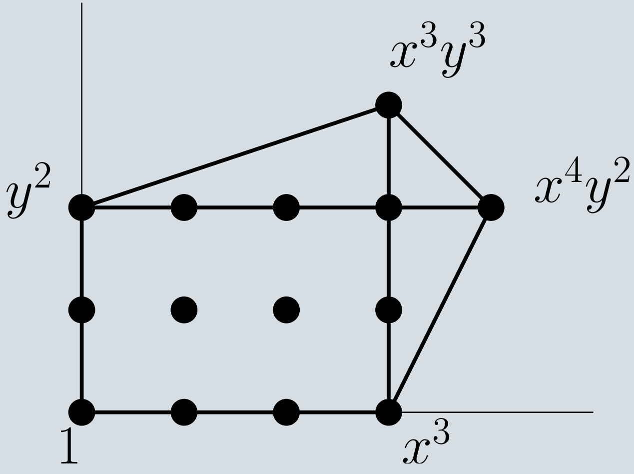
Theorem (Cools-D-Payne-Robeva)
B's starting position $\rightsquigarrow$ lingering lattice path in $\mathbb{Z}^r$;
B wins iff path stays in chamber $\{(x_1, \ldots, x_r) \mid x_1 > x_2 > \ldots > x_r > 0\}$.

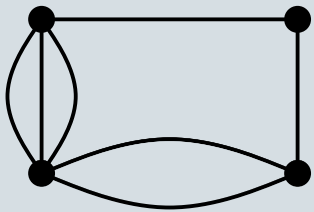
$\rightsquigarrow \rho \geq 0 \Leftrightarrow$ B wins

Castryck and Cools's gonality conjecture

$r = 1$
$f \in \mathbb{C}[x, y]$ general with Newton polytope Δ
$X := \{f = 0\}$ Riemann surface

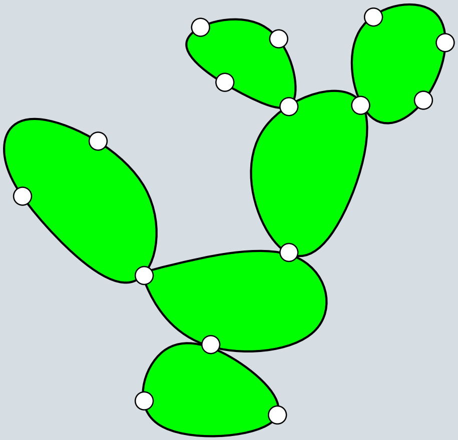
Conjecture
minimal d for which B wins on $X := \{f = 0\}$
(=minimal degree of a meromorphic map to $\mathbb{P}^1$)
equals $d =$ *lattice width* of Δ
(with two exceptions)





Purely combinatorial?

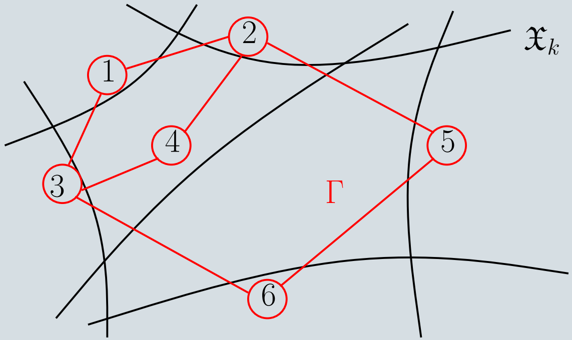
Theorem (van der Pol)
$\rho \geq 0$ and Γ a *cactus graph*
$\Rightarrow$ B has winning positions with all chips at vertices.



Future goal:
Understand Kleiman-Laksov for (metric) graphs.

Baker's Specialisation Lemma

$\mathfrak{X}$ *curve family* over $\mathbb{C}[[t]]$
(proper, flat, regular scheme)
generic fibre $\mathfrak{X}_{\mathbb{C}((t))}$ smooth curve X
special fibre $\mathfrak{X}_{\mathbb{C}} = X_1 \cup \ldots \cup X_s$
X_i smooth, intersections simple nodes
$\rightsquigarrow$ *dual graph* Γ on $\{u_1, \ldots, u_s\}$
(metric with edge lengths 1)
$\rightsquigarrow$ map $X(\mathbb{C}((t))) \to \{u_1, \ldots, u_s\}$



well-behaved with respect to finite extensions $\mathbb{C}((t^{1/n}))/\mathbb{C}((t))$
$\rightsquigarrow$ specialisation map $\tau : X(\mathbb{C}\{\{t\}\}) \to \Gamma_{\mathbb{Q}}$

Theorem
Brill wins with starting positing D on $X(\mathbb{C}\{\{t\}\})$
$\Rightarrow$ Baker wins with starting position $\tau(D)$ on $\Gamma_{\mathbb{Q}}$

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