Torus actions and faithful tropicalisation

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Setting

$\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ *tropical numbers*

$v : K \to \mathbb{R}_\infty$ field valuation, k residue field

$X \subseteq \mathbb{A}^n_K$ closed subvariety with ideal $I = I(X)$

$\rightsquigarrow \operatorname{Trop}(X) \subseteq \mathbb{R}^n_\infty$ *tropical variety*

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Tilted polynomial ring

$\xi \in \mathbb{R}^n_\infty \rightsquigarrow K[x]_\xi := \{\sum_\alpha c_\alpha x^\alpha \mid v(c_\alpha) + \alpha \cdot \xi \geq 0\}$

has ideal $\{\sum_\alpha c_\alpha x^\alpha \mid v(c_\alpha) + \alpha \cdot \xi > 0\}$

Quotient is $k[y_i \mid \xi_i \neq \infty]$ where y_i is image of $c_i x_i$, $v(c_i) + \xi_i = 0$.

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Initial ideal

$\mathrm{in}_\xi(I) :=$ image of $K[x]_\xi \cap I$ in $k[y_i \mid \xi_i \neq \infty]$

$\mathrm{Trop}(X) = \{\xi \in \mathbb{R}^n_\infty \mid \mathrm{in}_\xi I$ does not contain any monomial$\}$

Berkovich space

$X^{\mathrm{an}} := \{w : K[X] \to \mathbb{R}_\infty \mid w \text{ ring valuation extending } v\}$

Topology: $w_1, w_2, \ldots$ converge iff $(w_i(f))_i$ converges for all f.

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Fundamental theorem of tropical geometry

$\pi_X : X^{\mathrm{an}} \to \mathbb{R}^n_\infty, \quad w \mapsto (w(x_1), \ldots, w(x_n))$ maps onto $\mathrm{Trop}(X)$.

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When does π_X have a continuous section $\sigma_X : \mathrm{Trop}(X) \to X^{\mathrm{an}}$?

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Cueto-Häbich-Werner (2013)
For $X = \mathbf{Gr}(2, m) \subseteq \mathbb{A}^{m(m-1)/2}$ a section exists (& all mults are 1).

Linear spaces

Theorem

If $Y \subseteq \mathbb{A}^n$ is a linear subspace, then $\pi_Y : Y^{\mathrm{an}} \to \mathrm{Trop}(Y)$ has a continuous section.

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Construction (*constant coefficient case*)

1. pick $\eta \in \mathrm{Trop}(Y) \cap \mathbb{R}^n$
2. choose $\pi \in \mathrm{Sym}(n)$ such that $\eta_{\pi_1} \geq \cdots \geq \eta_{\pi_n}$
3. set $J_0 := \emptyset$ and

$J_i := J_{i-1} \cup \{\pi(i)\}$ if $x_{\pi(i)}|_Y \notin \langle x_j|_Y, j \in J_{i-1}\rangle$;
$\quad := J_{i-1}$ otherwise

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4. set $J := J_n$, a maximal-weight basis of the matroid defined by Y
5. pick $f \in K[Y] \rightsquigarrow$ unique expression $f = \sum_{\alpha \in \mathbb{N}^J} c_\alpha x^\alpha$
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(*extends continuously to points with* ∞ *coordinates*) □

Example: graphical matroid of K_m.

$Y \subseteq \mathbb{A}^{m(m-1)/2}$ defined by parameterisation $x_{ij} = (y_i - y_j)_{i<j}$

$\{x_{ij} \mid (i, j) \in J\}$ independent on Y iff J a tree

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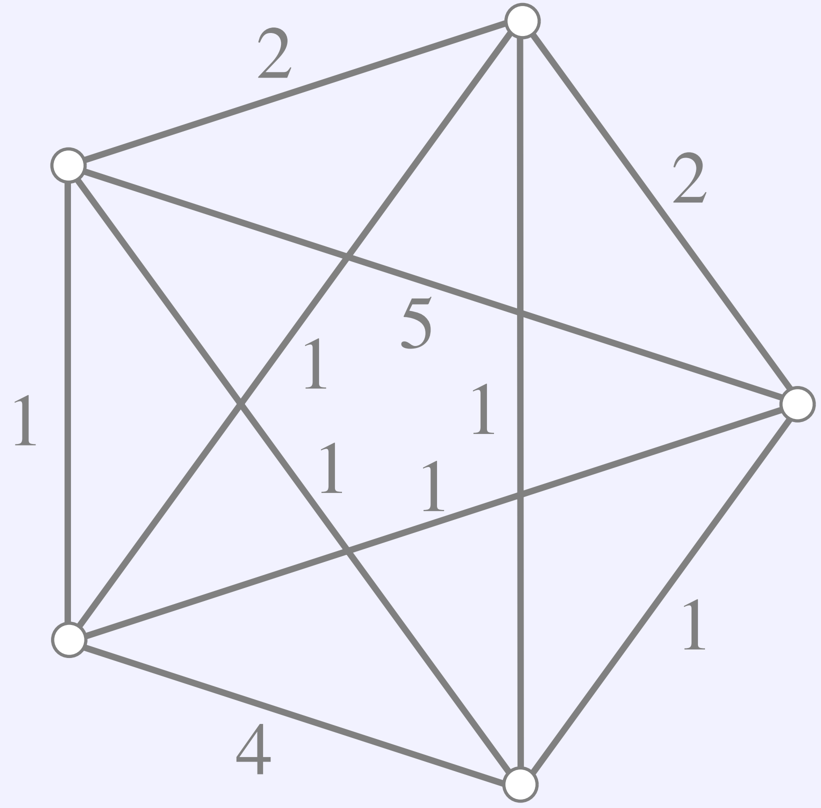

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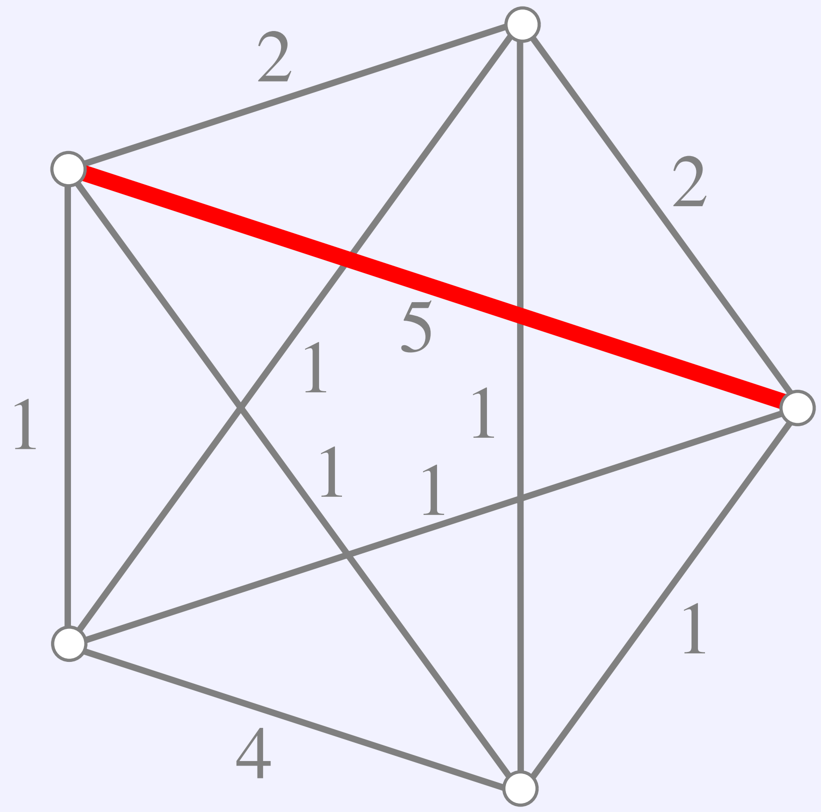

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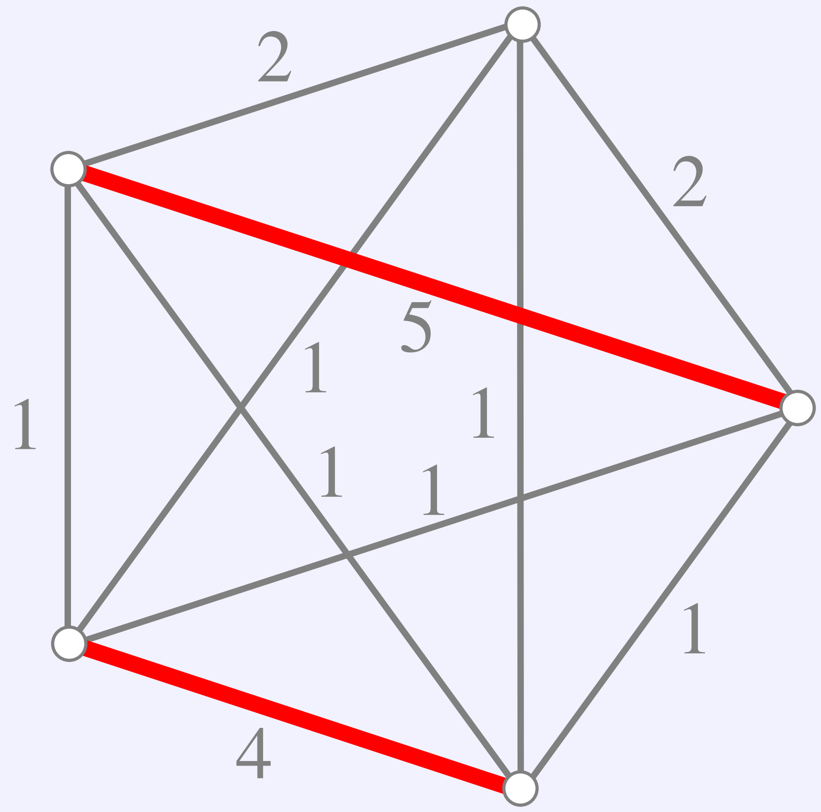

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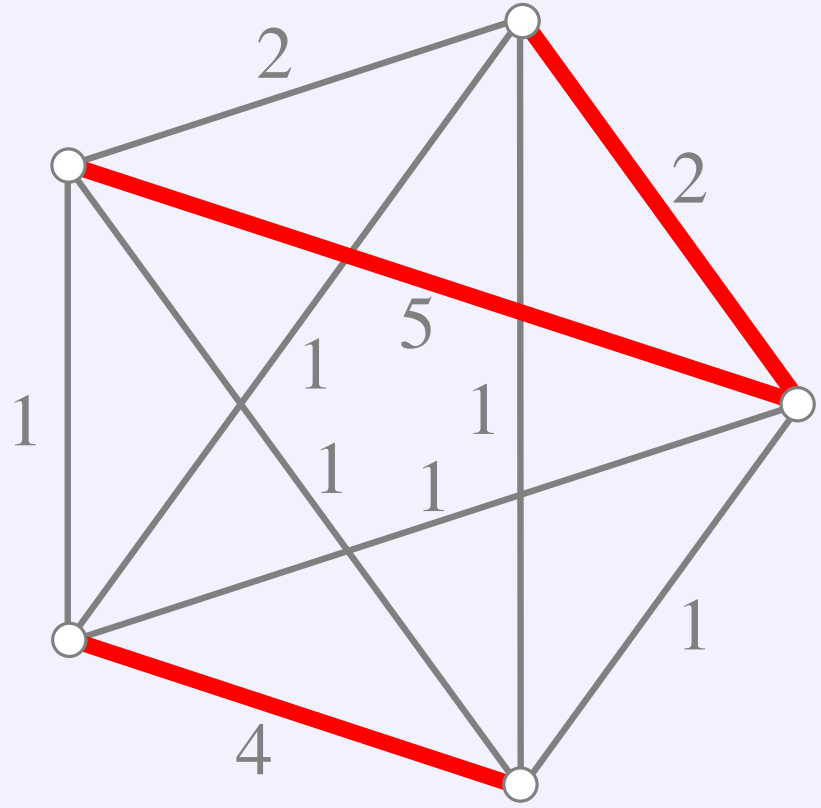

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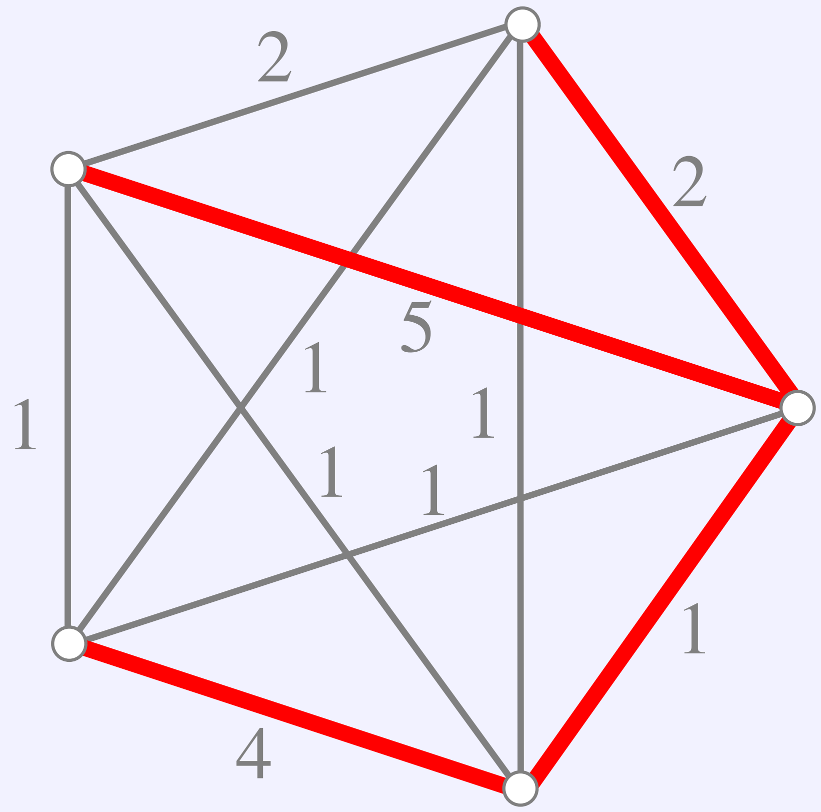

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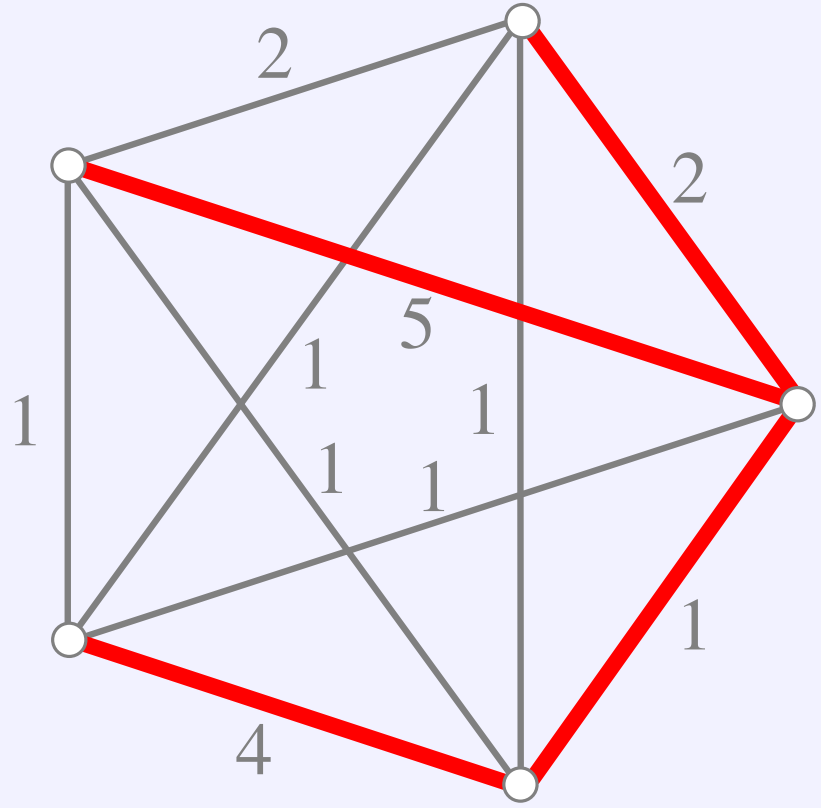

$\eta \in \mathrm{Trop}(Y)$

For $(i, j) \notin J$, η_{ij} is minimal weight in cycle closed by ij.
$\rightsquigarrow \sigma_Y(\eta)(x_{ij}) = \eta_{ij}$

Torus actions

Setting

$X \subseteq \mathbb{A}^n$

$\varphi : \mathbb{G}_\mathrm{m}^m \to \mathbb{G}_\mathrm{m}^n$ torus homomorphism given by $A \in \mathbb{Z}^{n \times m}$

assume X stable under $\varphi(\mathbb{G}_\mathrm{m}^m)$

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Well-known fact

$\mathbb{R}^m$ acts on $\mathrm{Trop}(X)$ by $(\tau, \xi) \mapsto A\tau + \xi$.

($A\mathbb{R}^m$ is contained in *lineality space* of $\mathrm{Trop}(X)$)

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Definition of μ

1. pick $\tau \in \mathbb{R}^m$, $w \in X^{\mathrm{an}}$, $f \in K[X]$
2. write $f(\varphi(t)x) = \sum_{\beta \in \mathbb{Z}^m} t^\beta f_\beta(x)$
3. $\mu(\tau, w)(f) := \min_\beta (w(f_\beta) + \beta \cdot \tau)$

Retract

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Example

$X = \mathbb{A}^2, \varphi : \mathbb{G}_{\mathrm{m}}^1 \to \mathbb{G}_{\mathrm{m}}^2, t \mapsto (t, t^{-1})$

$f = \sum_{ij} c_{ij} x^i y^j \rightsquigarrow \mu(\tau, w)(f) = \min_{k \in \mathbb{Z}}(w(\sum_{i-j=k} c_{ij} x^i y^j) + k\tau)$

In general $\mu(0, w) \neq w$*!*

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Lemma

1. $\mu(\tau_1 + \tau_2, w) = \mu(\tau_1, \mu(\tau_2, w))$
2. Z :=image(μ) is a retract of X^{an} on which $\mathbb{R}^m$ acts continuously.

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\mathbb{R}^m \times Z & \xrightarrow{\ \text{action}\ } & Z \\
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$Y \subseteq \mathbb{A}^n$ linear subspace

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$X := \overline{\varphi(\mathbb{G}_\mathrm{m}^m)Y}$; $X^0 := X \cap \mathbb{G}_\mathrm{m}^n$

Smearing around a subspace by a torus

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Proposition

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Smearing around a subspace by a torus

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I don't know a general condition for extending to all of $\mathrm{Trop}(X)$.

Example 1: maximal minors

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$X \subseteq \mathbb{A}^{m\times p}, m \leq p$ defined by $m \times m$-minors
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3. $\mathbb{R}^m$ acts by translation on hyperplanes.
4. Pick $\xi \in \mathrm{Trop}(X^0)$
5. $-\tau :=$ stable intersection of hyperplanes of first $m-1$ columns
6. $\eta := -A\tau + \xi$

Example 1: maximal minors

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$X \subseteq \mathbb{A}^{m\times p}, m \leq p$ defined by $m \times m$-minors
$\rightsquigarrow$ the map $(X^0)^{\text{an}} \to \text{Trop}(X^0)$ has an $\mathbb{R}^m$-equivariant section.

1. $X = \overline{\mathbb{G}_{\text{m}}^m Y}$ where $Y = \{x \mid (1,\ldots,1)x = 0\}$
2. $\eta \in \mathbb{R}^{m\times p}$ lies in $\text{Trop}(Y^0)$ iff $(0,\ldots,0)$ lies in tropical hyperplane $\subseteq \mathbb{R}^m$ defined by each column.
3. $\mathbb{R}^m$ acts by translation on hyperplanes.
4. Pick $\xi \in \text{Trop}(X^0)$
5. $-\tau :=$ stable intersection of hyperplanes of first $m-1$ columns
6. $\eta := -A\tau + \xi$
7. $\text{Trop}(X^0) \to \mathbb{R}^m \times \text{Trop}(Y^0),\ \xi \mapsto (\tau,\eta)$ is continuous.
8. Section: $\text{Trop}(X^0) \to \mathbb{R}^m \times \text{Trop}(Y^0) \to \mathbb{R}^m \times (Y^0)^{\text{an}} \to Z$ $\square$

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(*I don't know if this extends to* Trop(X).)

Example 2: Grassmannians of planes

New proof of Cueto, Häbich, Werner (2013)
For $X = \mathbf{Gr}(2, n) \subseteq \mathbb{A}^{m(m-1)/2}$ the map $X^{\mathrm{an}} \to \mathrm{Trop}(X)$ has a continuous section.

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New proof of Cueto, Häbich, Werner (2013)
For $X = \mathbf{Gr}(2,n) \subseteq \mathbb{A}^{m(m-1)/2}$ the map $X^{\mathrm{an}} \to \mathrm{Trop}(X)$ has a continuous section.

1. X is image of $\mathbb{A}^m \times \mathbb{A}^m \to \mathbb{A}^{m(m-1)/2}, (y,z) \mapsto (y_i z_j - y_j z_i)_{i<j}$
2. $Y \subseteq X$ obtained by setting $z := (1,\ldots,1)$
3. $X^0 = \mathbb{G}_{\mathrm{m}}^m Y$
4. $\mathrm{Trop}(X^0)$ parameterises tropical lines in $(\mathbb{R}_\infty^m - \infty)/\mathbb{R}(1,\ldots,1)$
5. $\mathrm{Trop}(Y)$ parameterises lines through $(0,\ldots,0)$.
6. Fix a tropical hyperplane H.
7. For $\xi \in \mathrm{Trop}(X^0)$, let $-\tau$ be stable intersection of line with H.
8. Set $\eta := -A\tau + \xi \in \mathrm{Trop}(Y)$
9. Set $\sigma_X(\xi) := \mu(\tau, \sigma_Y(\eta))$ (*uses graphical matroid of* K_m)
10. Show that it extends. □

Example 2, continued

Remark

In this case, the map $\mathbb{R}^m \times \mathrm{Trop}(Y^0) \to Z,\ (\tau, \eta) \mapsto \mu(\tau, \sigma_Y(\eta))$ *factorises* through $\mathbb{R}^m \times \mathrm{Trop}(Y^0)$, i.e., decomposition of ξ as $A\tau + \eta$ is irrelevant. This is used in the proof, and implies $\mathbb{R}^m$-equivariance.

Example 3: rank-two matrices.

Theorem
$X \subseteq \mathbb{A}^{m\times p}$ variety defined by 3×3-minors
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1. as for $\mathbf{Gr}(2, m)$, now using $Y = (y_i - z_j)_{ij}$
$\rightsquigarrow$ graphical matroid of $K_{m,p}$.
2. can make it $\mathbb{R}^m$-equivariant or $\mathbb{R}^p$-equivariant, but (maybe) not both.
3. uses work by Develin-Santos-Sturmfels.

Setting

$\varphi : \mathbb{G}_{\mathrm{m}}^m \to \mathbb{G}_{\mathrm{m}}^n$

$V = \overline{\varphi(\mathbb{G}_{\mathrm{m}}^m)} \subseteq \mathbb{A}^n$ toric variety

$X = \{h \in \mathbb{A}^n \mid h \perp T_p V \text{ for some } p \in V\}$ dual variety

$Y = (T_{\varphi(1)} V)^{\perp}$ linear subspace

$\rightsquigarrow X = \overline{\varphi(\mathbb{G}_{\mathrm{m}}^m) Y}$ (*Horn uniformisation*)

A-discriminants

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Dickenstein-Feichtner-Sturmfels (2007)

$\mathrm{Trop}(X^0) = A\mathbb{R}^m + \mathrm{Trop}(Y^0)$

(*and a ray shooting method for computing* $\mathrm{Trop}(X^0)$)

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Question

Does there exist a section $\mathrm{Trop}(X^0) \to Z^0$, or even $\mathrm{Trop}(X) \to Z$?

Example 4: Cayley's hyperdeterminant

$V \subseteq \mathbb{A}^{2\times 2\times 2}$ rank-one tensors
X =dual variety, a quartic hyperplane

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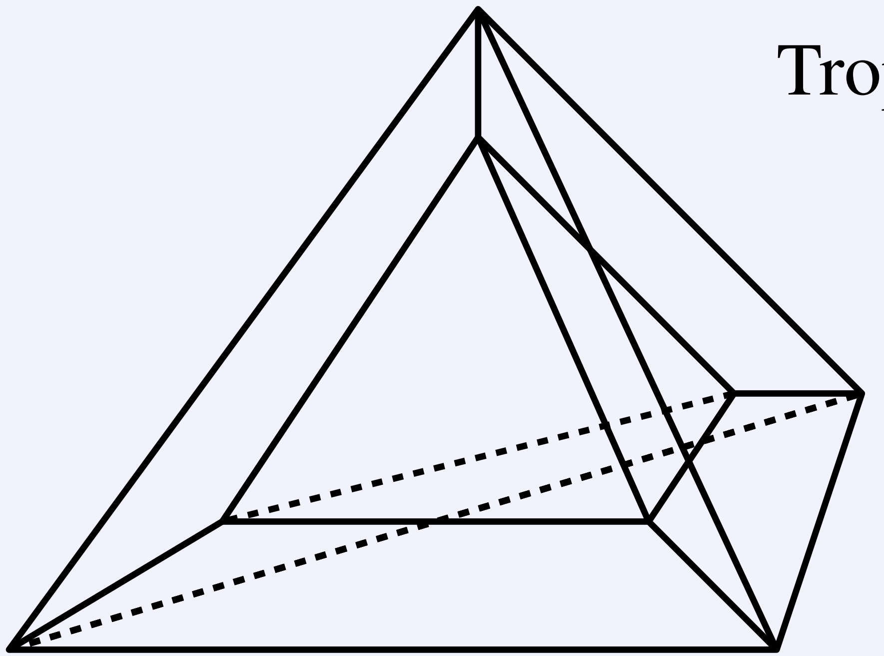

Trop(X)/$A\mathbb{R}^{2+2+2}$ intersected with a S^3

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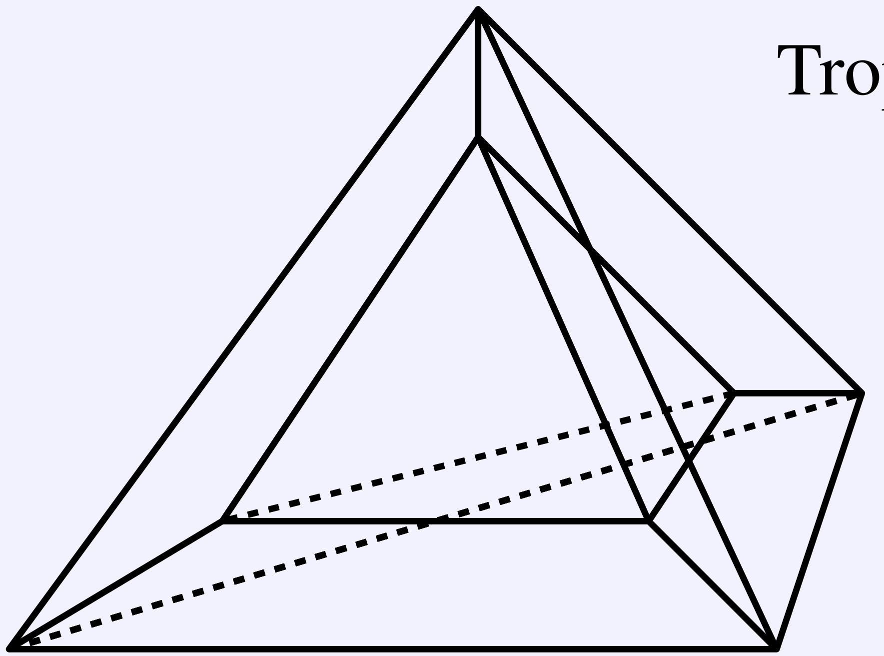

$\mathrm{Trop}(X)/A\mathbb{R}^{2+2+2}$ intersected with a S^3

Theorem

$(X^0)^{\mathrm{an}} \supseteq Z^0 \to \mathrm{Trop}(X^0)$ has a continuous $\mathbb{R}^{2+2+2}$-equivariant section $\rightsquigarrow$ the double tetrahedron is a retract of $(X^0/\mathbb{G}_m^{2+2+2})^{\mathrm{an}}$.