The algebra of symmetric high-dimensional data

Jan Draisma
TU Eindhoven

GeorgiaTech, April 2014

The algebra of symmetric high-dimensional data

·	3	0	3	1	3	0	3	6	·
·	6	0	6	2	6	0	1	2	·

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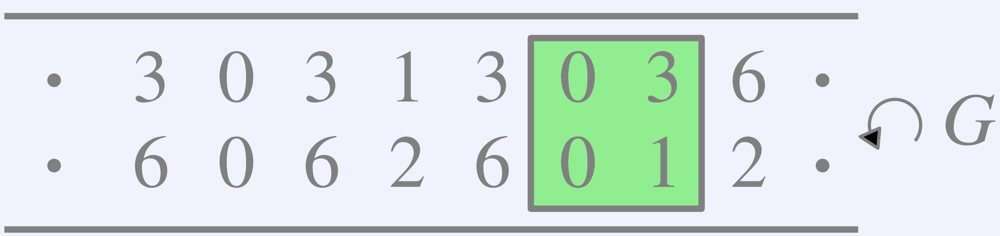
What global data properties can be tested locally?



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$\circlearrowleft G$

rank one?

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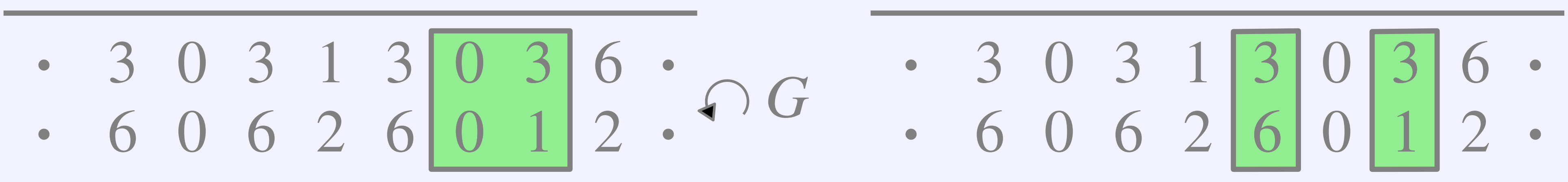

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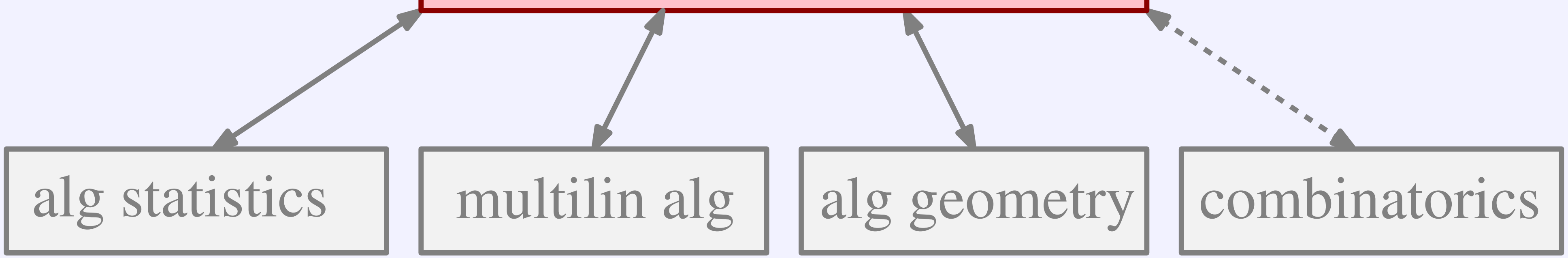

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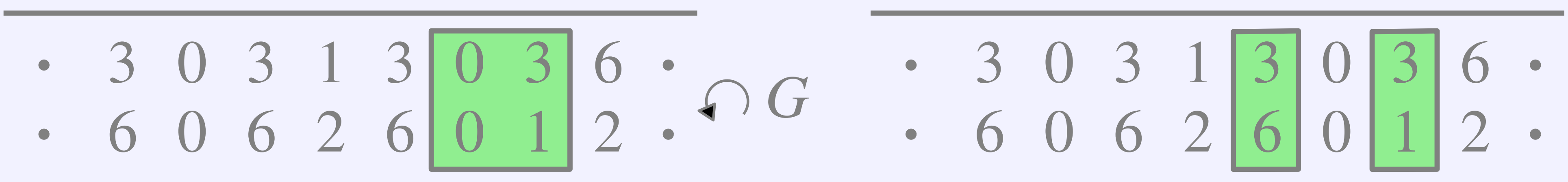

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History: Hilbert's Basis Theorem

David Hilbert

Any polynomial system

$f_1(x_1, \ldots, x_n) = 0,$

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reduces to a *finite* system

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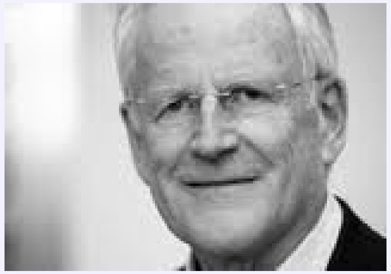
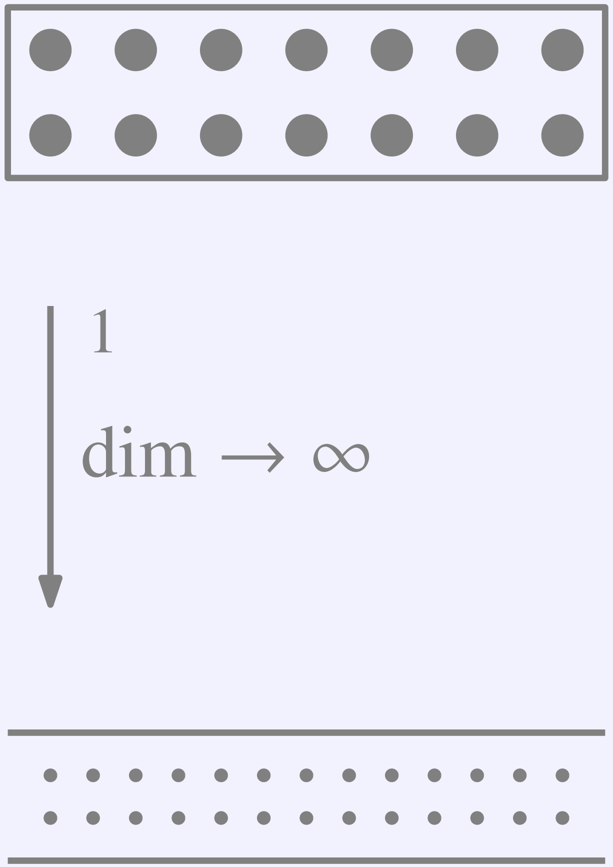
Bruno Buchberger
Gröbner bases, algorithmic methods



1. Model

high-dim data $\rightsquigarrow$ ∞-dim data space
data property $\rightsquigarrow$ ∞-dim subvariety
small window $\rightsquigarrow$ finite window
$\rightsquigarrow$ *leave fin-dim commutative algebra*



Three-step approach

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∞-dim property finitely defined

up to symmetry?

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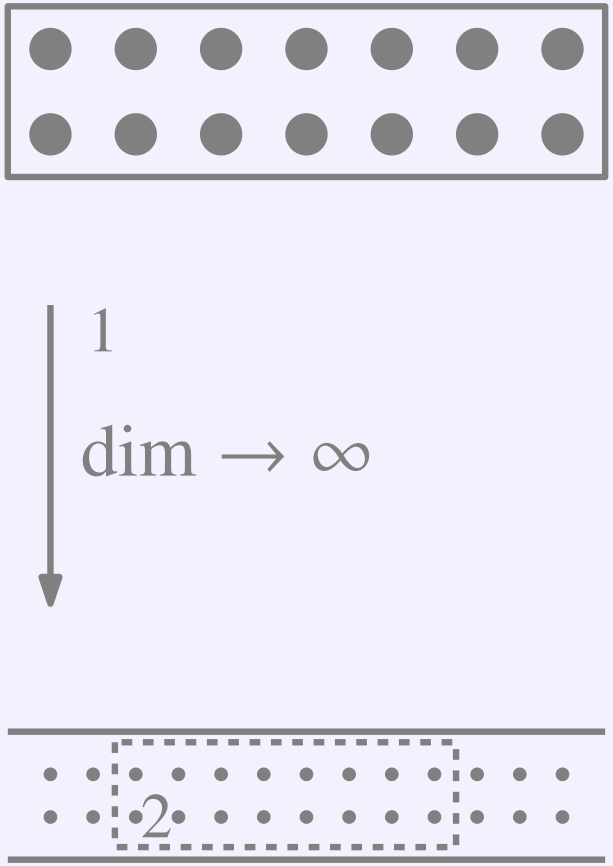

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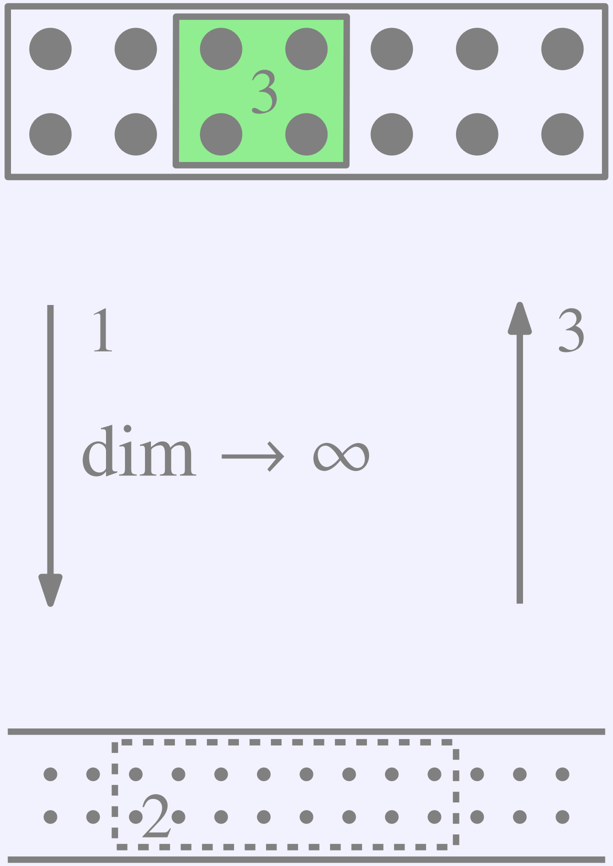
3. Compute

actual windows for fin-dim data
⇝ *generalise Buchberger alg to ∞ variables*



Noetherianity up to symmetry

Example

$K[x_1, x_2, \ldots]$ is not Noetherian, e.g. $x_1 = 0, x_2 = 0, \ldots$
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Cohen [*J Algebra, 1967*]

$K[x_1, x_2, \ldots]$ *is* Sym($\mathbb{N}$)-Noetherian, i.e., every Sym($\mathbb{N}$)-stable system reduces to finitely many equations up to Sym($\mathbb{N}$).

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Fundamental (non-)examples

$K[x_{ij} \mid i \in \{1, \ldots, k\}, j \in \mathbb{N}]$ is Sym($\mathbb{N}$)-Noetherian;
$K[x_{ij} \mid i, j \in \mathbb{N}]$ is *not* Sym($\mathbb{N}$)-Noetherian, *but* it *is* $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian, and so is $(K^{\mathbb{N}\times\mathbb{N}})^p$ for all p.

Noetherianity up to symmetry, continued

Hillar-Sullivant [*Adv Math, 2012*]

$S := K[x_{ij} \mid i \in \{1, \ldots, k\}, j \in \mathbb{N}]$ is $\mathrm{Sym}(\mathbb{N})$-Noetherian.

Noetherianity up to symmetry, continued

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Open question

Given $\varphi : R \to S$ reasonable Sym($\mathbb{N}$)-equivariant map,
is ker(φ) generated by finitely many Sym($\mathbb{N}$)-orbits?

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Yes if R is a polynomial ring with finitely many Sym($\mathbb{N}$)-orbits of variables and moreover φ is *monomial*.

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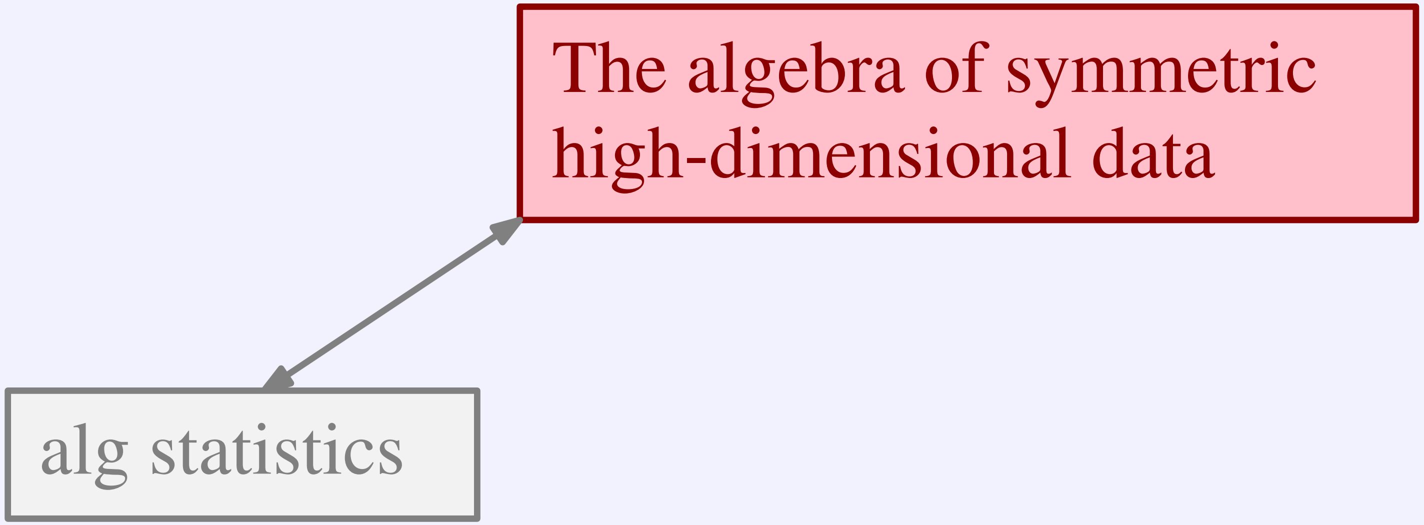
Moreover, Krone-Leykin have developed and implemented an ∞-dimensional Buchberger algorithm for computing $\ker \varphi$.


The algebra of symmetric high-dimensional data
alg statistics

Setting

$X_1, \ldots, X_n$ jointly Gaussian, mean 0

$\rightsquigarrow$ explained well by $k \ll n$ *factors*?

i.e., is $X_i = \sum_j s_{ij} Z_k + t_i \epsilon_i$, with $Z_1, \ldots, Z_k$, $\epsilon_1, \ldots, \epsilon_n$ independent standard normals?

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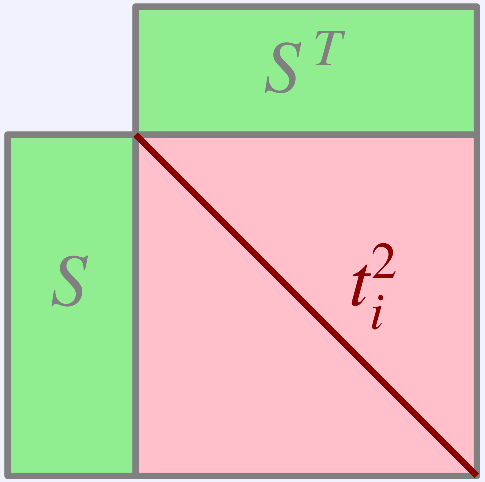

Gaussian factor analysis

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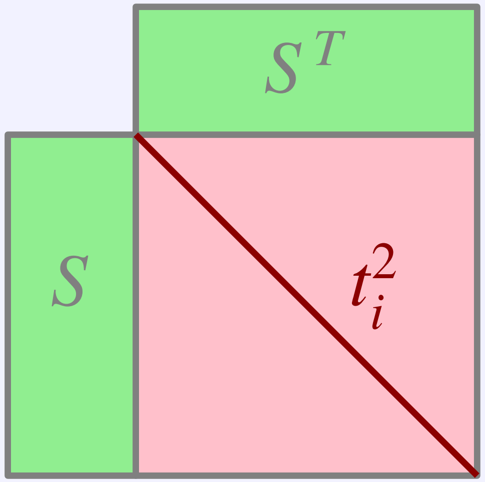

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Definition

$F_{k,n} := \overline{\{S S^T + \text{diag}(t_1^2, \ldots, t_n^2) \mid S \in \mathbb{R}^{n \times k}, t_i \in \mathbb{R}\}}$

$\rightsquigarrow$ *algebraic variety in* $\mathbb{R}^{n \times n}$ *called Gaussian k-factor model*

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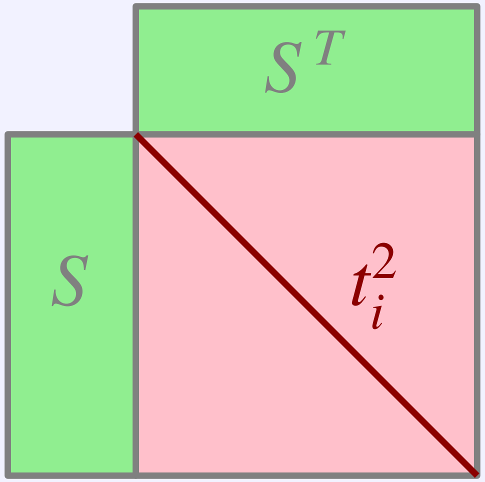

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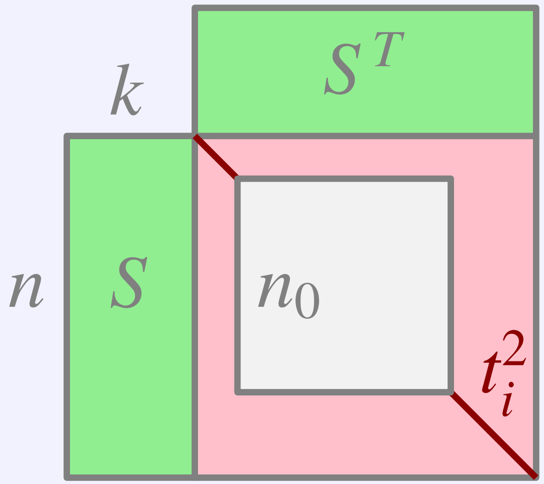
Example

$F_{2,5}$ is zero set of $\{\sigma_{ij} - \sigma_{ji} \mid i, j = 1, \ldots, 5\}$ and the *pentad*
$\sum_{\pi \in \mathrm{Sym}(5)} \operatorname{sgn}(\pi) \sigma_{\pi(1)\pi(2)} \sigma_{\pi(2)\pi(3)} \sigma_{\pi(3)\pi(4)} \sigma_{\pi(4)\pi(5)} \sigma_{\pi(5)\pi(1)}$

Drton-Sturmfels-Sullivant [*Prob Th Rel Fields 2007*]

If $\Sigma \in F_{k,n}$ then any principal $n_0 \times n_0$ submatrix $\Sigma' \in F_{k,n_0}$.

⇝ *Is there an* $n_0 = n_0(k)$ *such that the converse holds for* $n \geq n_0$*?*



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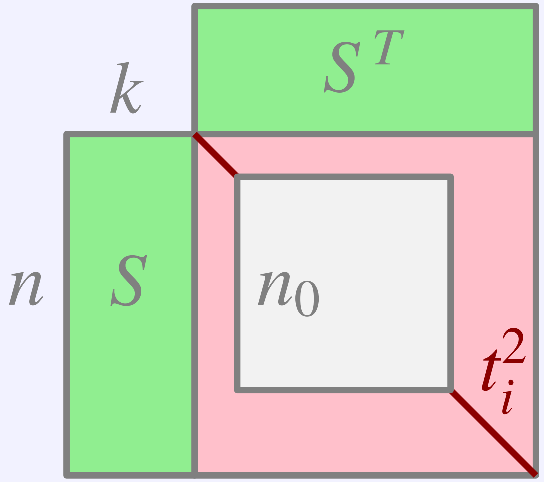

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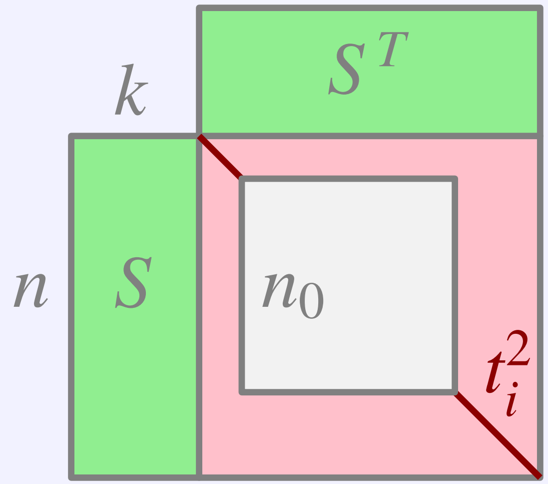
yes for all k ($n_0 =$?)

$\rightsquigarrow$ *uses* $F_{k,\infty}$ *and Noetherianity up to* $\mathrm{Sym}(\mathbb{N})$



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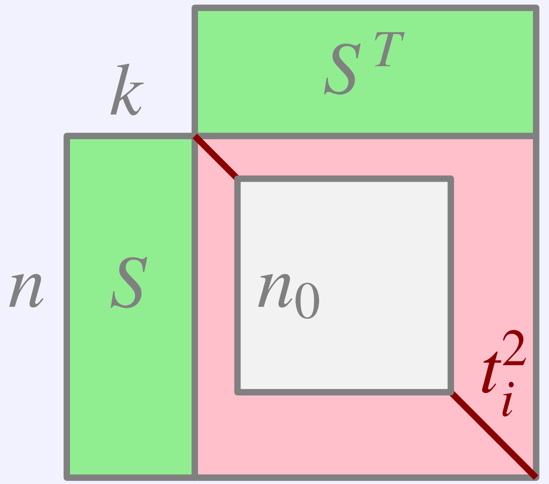
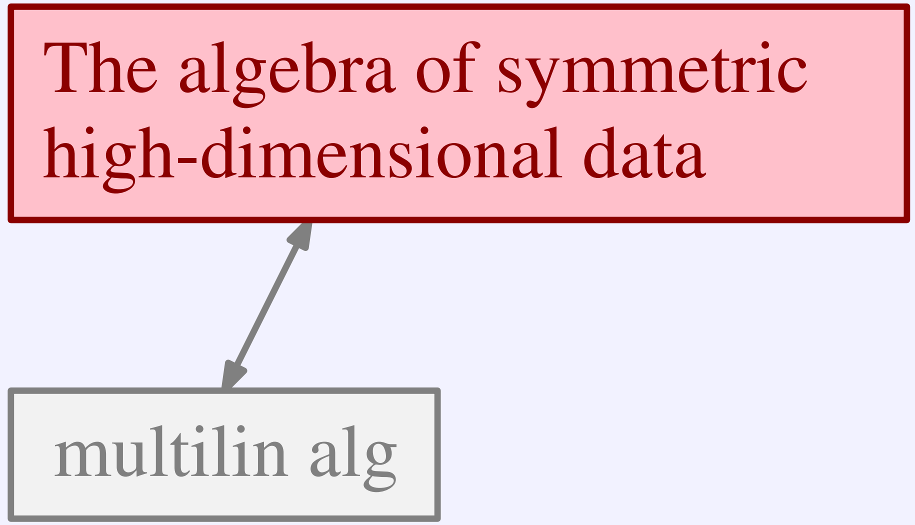


Brouwer-Draisma [*Math Comp 2011*]

yes for $k = 2$: pentads and 3×3-minors define $F_{2,n}$, $n \geq n_0 := 6$

$\rightsquigarrow$ *uses* Sym($\mathbb{N}$)*-Buchberger algorithm (+ a weekend on 20 computers)*

$\rightsquigarrow$ *a* **single** *computation proves this* **for all** *n*


The algebra of symmetric high-dimensional data
multilin alg

Tensor rank

A wrong-titled movie

tensor T=multi-indexed array of numbers

matrices=two-way tensors

this picture=three-way tensor, . . .

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Pure tensor P
has entries $P_{i,j,\ldots,k} = x_i y_j \cdots z_k$
for vectors $x, \ldots, z$
$\rightsquigarrow$ *for a matrix:* xy^T*, rank one*

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Tensor rank of T
is minimal k in $T = \sum_{j=1}^{k} P^{(j)}$ with each $P^{(j)}$ pure
⇝ *generalises matrix rank*
⇝ *useful for MRI data, communication complexity, phylogenetics etc.*

Finiteness for bounded-rank tensors

Matrix rank	**Tensor rank**
efficiently computable	NP-hard
field independent	field dependent
can only go down in limit	can also go up

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Draisma-Kuttler [*Duke 2014*]
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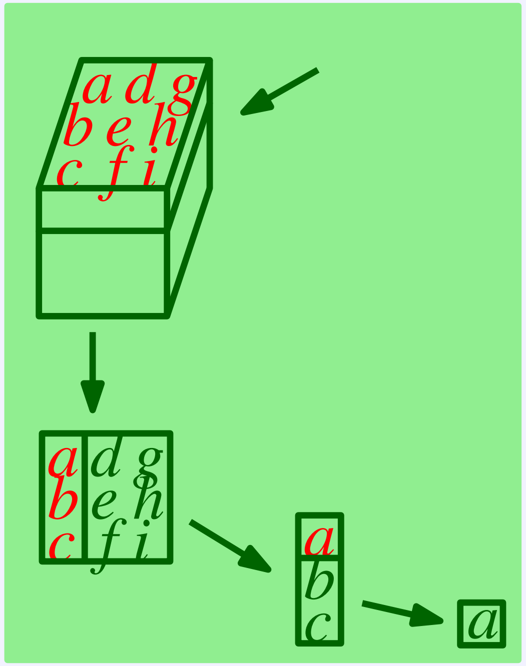

$\bigcup_{n=0}^{\infty} \mathrm{Sym}(n) \ltimes \mathrm{GL}_3^n$

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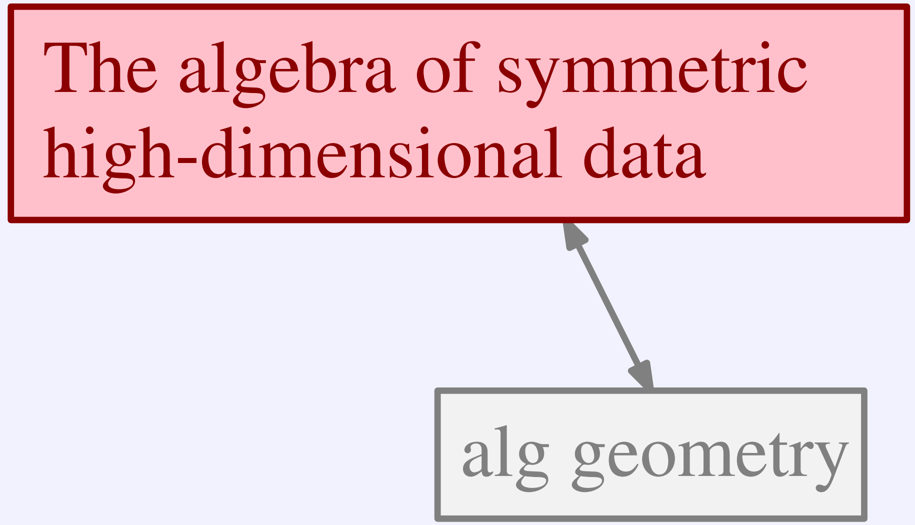
The algebra of symmetric high-dimensional data
alg geometry

Grassmannians: functoriality and duality

V a fin-dim vector space over an infinite field K
$\rightsquigarrow \mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
cone over Grassmannian
(*rank-one alternating tensors*)

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Two properties:
1. if $\varphi : V \to W$ linear
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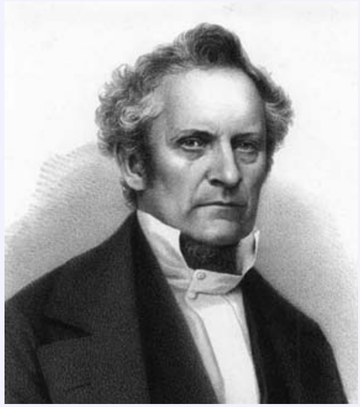
2. if $\dim V =: n + p$ with $n, p \geq 0$
$\rightsquigarrow$ natural map $\bigwedge^p V \to (\bigwedge^n V)^* \to \bigwedge^n(V^*)$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_n(V^*)$

Plücker varieties



Definition

Rules $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ with

$$\mathbf{X}_p : \{\text{vector spaces } V\} \to \{\text{varieties in } \textstyle\bigwedge^p V\}$$

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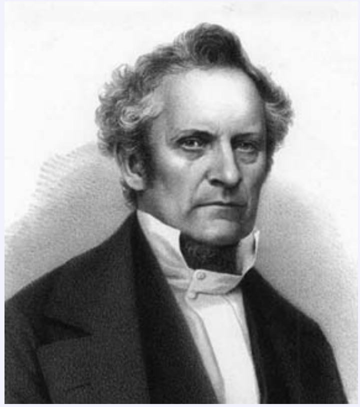

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form a *Plücker variety* if, for $\dim V = n + p$,
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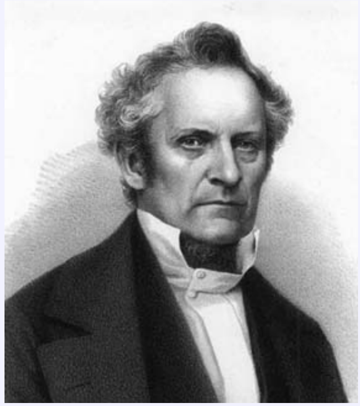
Plücker varieties



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$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are

$\mathbf{X} + \mathbf{Y}$ *(join)*, $\tau\mathbf{X}$ *(tangential)*,

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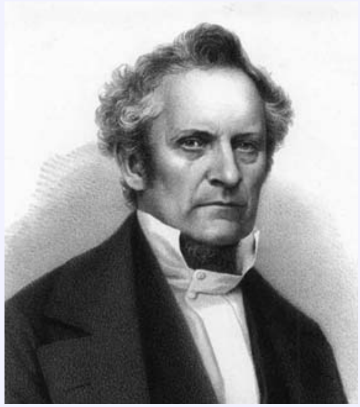

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skew analogue of Snowden's Δ-varieties

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Theorems apply, in particular, to $k\mathbf{Gr}$ = k-th *secant variety* of $\mathbf{Gr}$.

The infinite wedge

$$V_\infty := \langle \dots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \dots \rangle_K$$
$$V_{n,p} := \langle x_{-n}, \dots, x_{-1}, x_1, \dots, x_p \rangle \subseteq V_\infty$$

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Diagram

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$\bigwedge^{\infty/2} V_\infty := \lim_\rightarrow \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \ldots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

Recall

$\bigwedge^{\infty/2} V_\infty$ has basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, where
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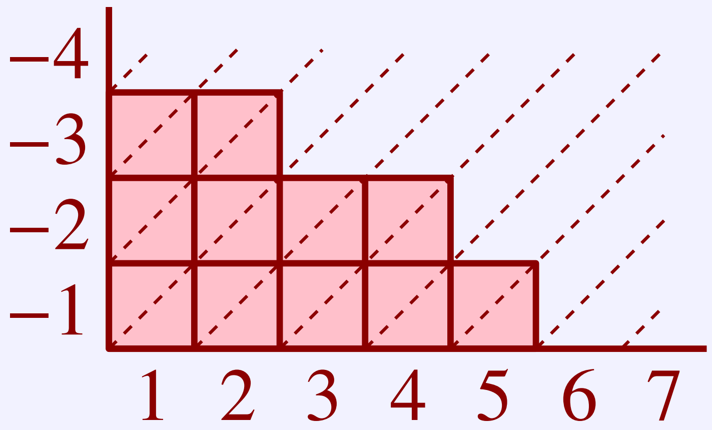
Bijection with Young diagrams

x_I with $I = \{-3, -2, 1, 2, 4, 6, 7, \ldots\}$ corresponds to



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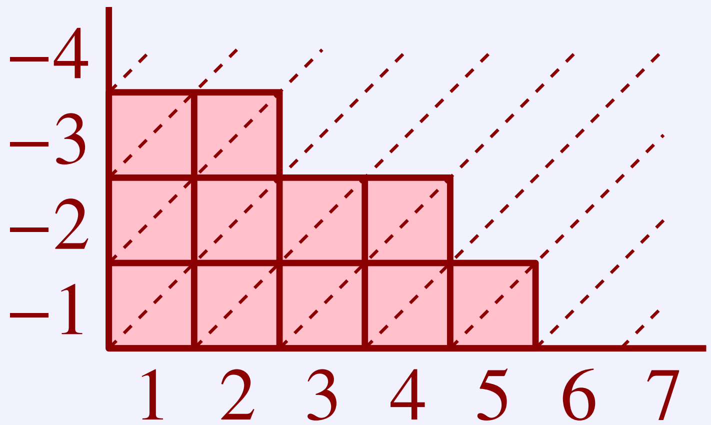

These x_I will be the *coordinates* of our ambient space, partially ordered by $I \leq J$ if $i_k \geq j_k$ for all k (inclusion of Young diags).
Unique minimum is $I = \{1, 2, \ldots\}$.

Dual diagram

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The limit of a Plücker variety

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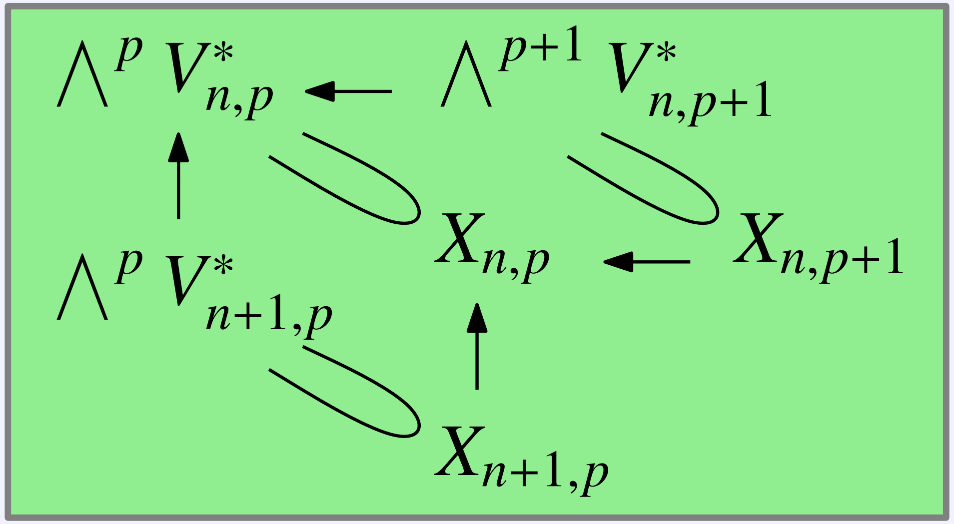

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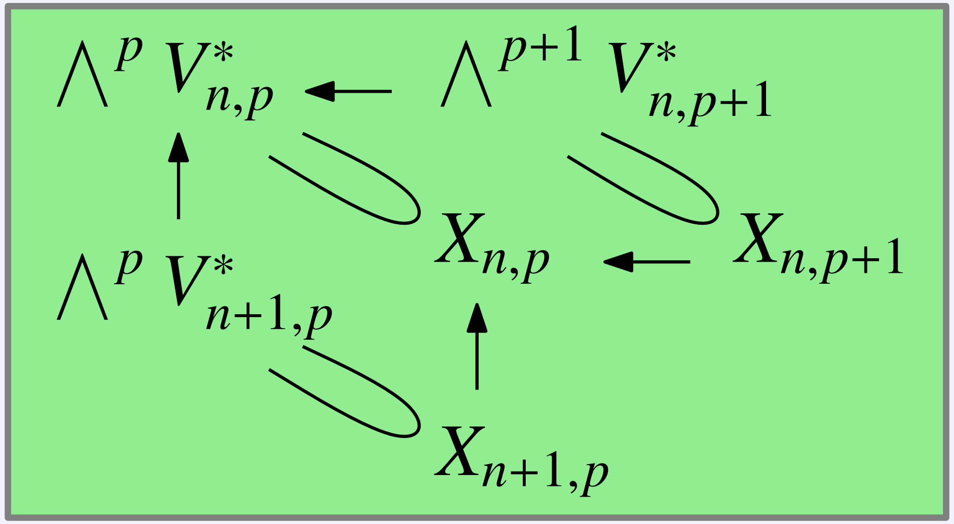

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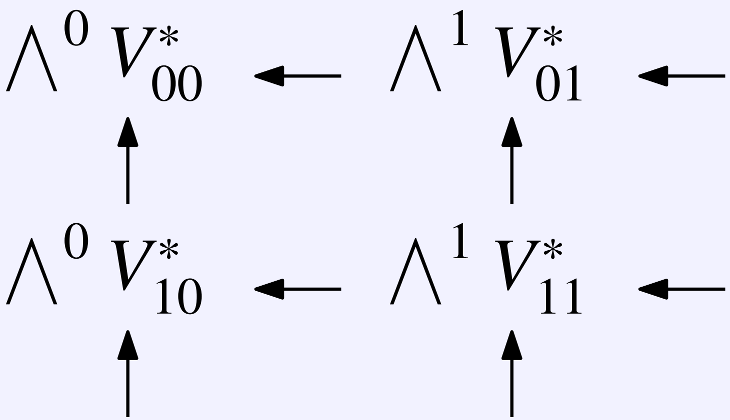

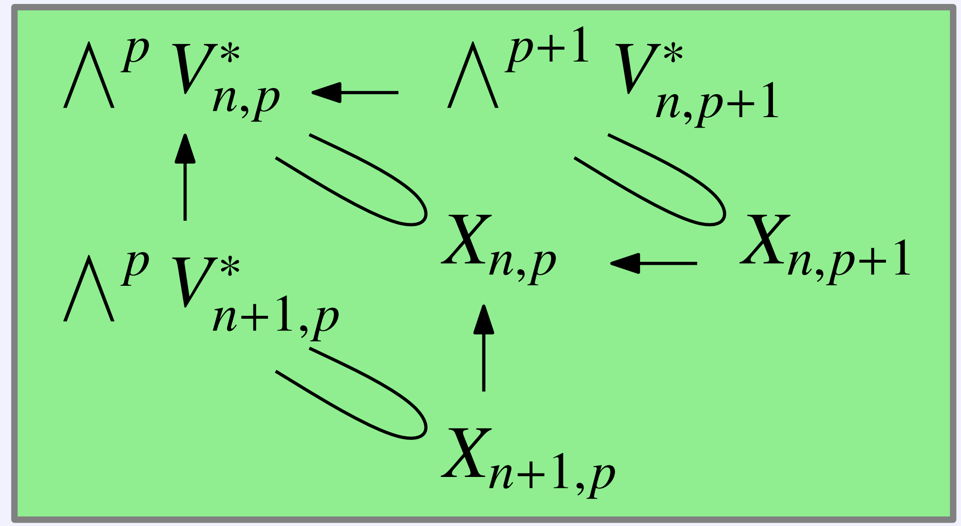

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Theorem (implies earlier)
For *bounded* $\mathbf{X}$, the limit $\mathbf{X}_\infty$ is cut out by finitely many GL_∞-orbits of equations.

Example
The limit $\mathbf{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$ of $(\mathbf{Gr}_p)_p$ is *Sato's Grassmannian* defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$ where $i_k = k - 1$ for $k \gg 0$ and $j_k = k + 1$ for $k \gg 0$.

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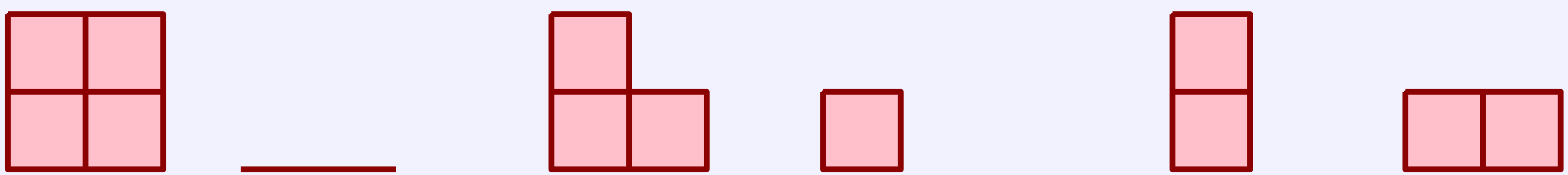
But in fact the GL_∞-orbit of

$(x_{-2,-1,3,\ldots} \cdot x_{1,2,3,\ldots}) - (x_{-2,1,3,\ldots} \cdot x_{-1,2,3,\ldots}) + (x_{-2,2,3,\ldots} \cdot x_{-1,1,3,\ldots})$



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Sato's Grassmannian

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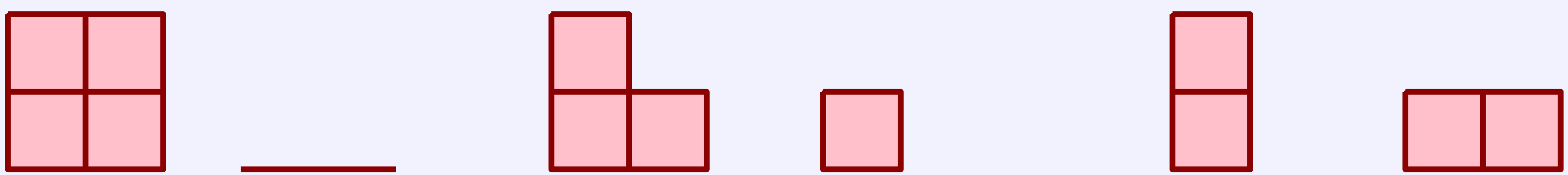

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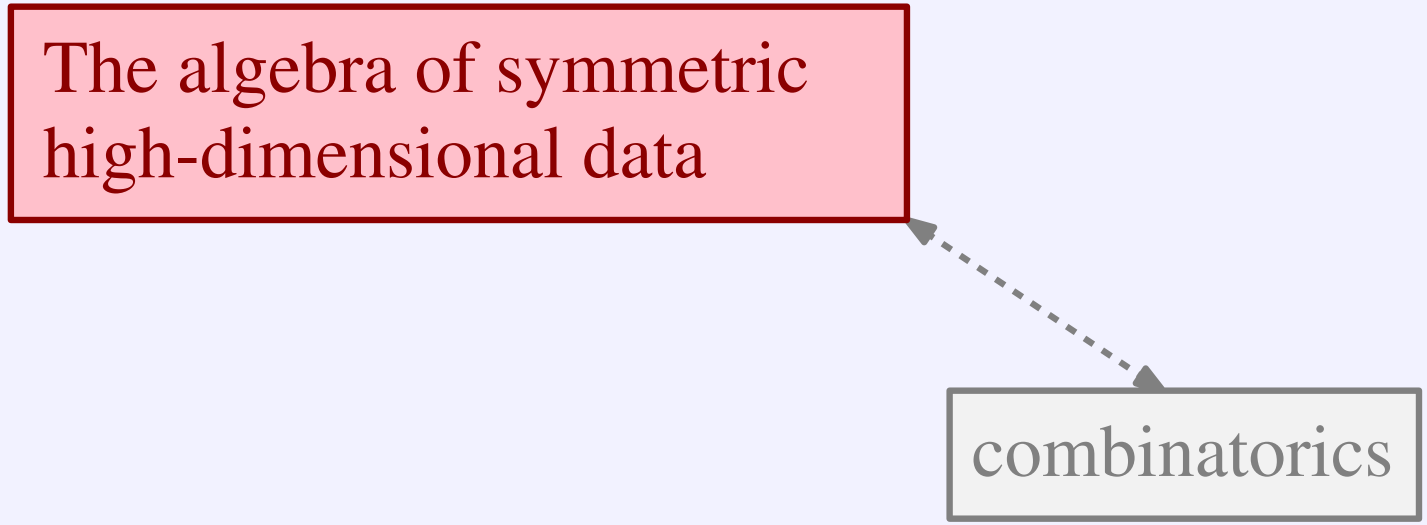
Our theorems imply that also higher secant varieties of Sato's Grassmannian are defined by finitely many GL_∞-orbits of equations. . . *we just don't know which!*

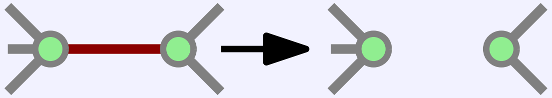
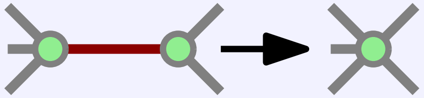


Conjecture

Over any field K, Sato's Grassmannian $\mathbf{Gr}_\infty(K)$ is Noetherian up to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N}) \subseteq \mathrm{GL}_\infty$.

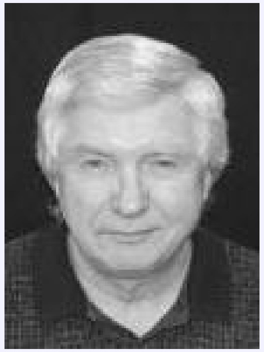
Graph minors

Any sequence of operations

 and 

takes a graph to a *minor*.

The graph minor theorem



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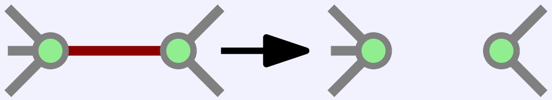
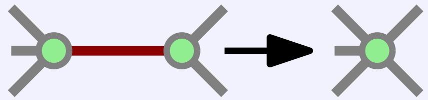
 and 

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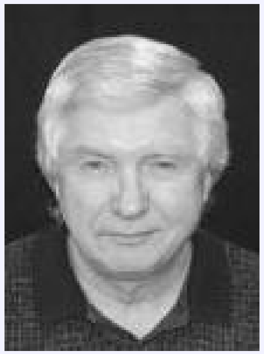
Robertson-Seymour [JCB 1983–2004, 669pp]

Any network property preserved under taking minors can be characterised by *finitely many forbidden minors*.



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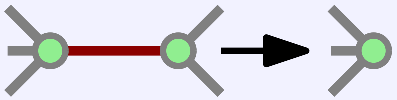
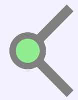
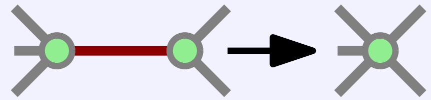
  and 

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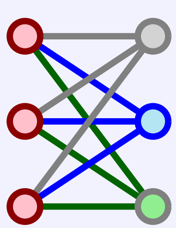
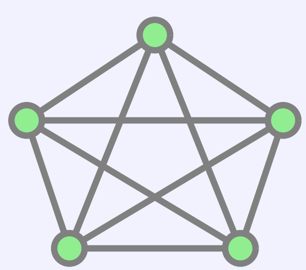
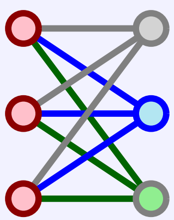
Wagner [Math Ann 1937]

For *planarity* these are

 and 

From graphs to matroids



1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
−1	−1	−1	0	0	0	0	0	0
0	0	0	−1	−1	−1	0	0	0
0	0	0	0	0	0	−1	−1	−1

column independence structure = matroid

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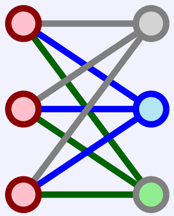
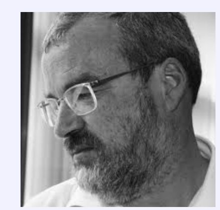
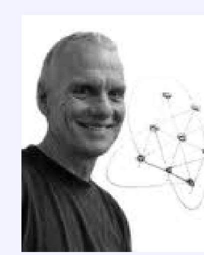


$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

column independence structure = matroid

Matroid minor theorem (Geelen-Gerards-Whittle)
Any minor-preserved property of matroids over a fixed *finite field K* can be characterised by finitely many forbidden minors.





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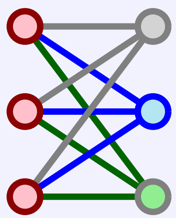
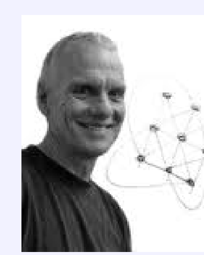


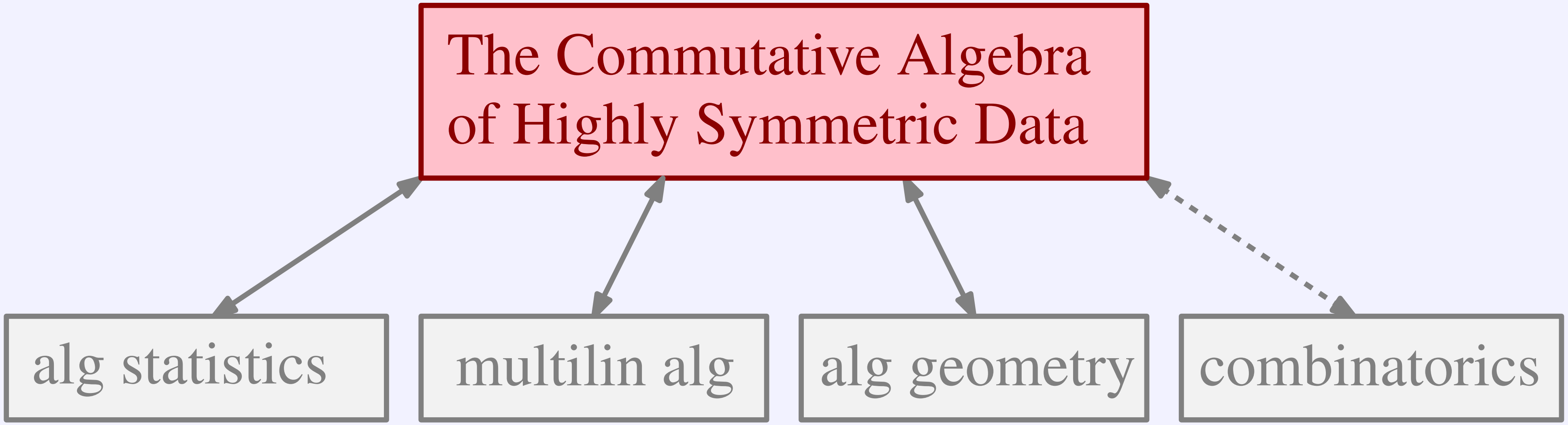
1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
−1	−1	−1	0	0	0	0	0	0
0	0	0	−1	−1	−1	0	0	0
0	0	0	0	0	0	−1	−1	−1

→ *column independence structure = matroid*

Matroid minor theorem (Geelen-Gerards-Whittle)
Any minor-preserved property of matroids over a fixed *finite field K* can be characterised by finitely many forbidden minors.

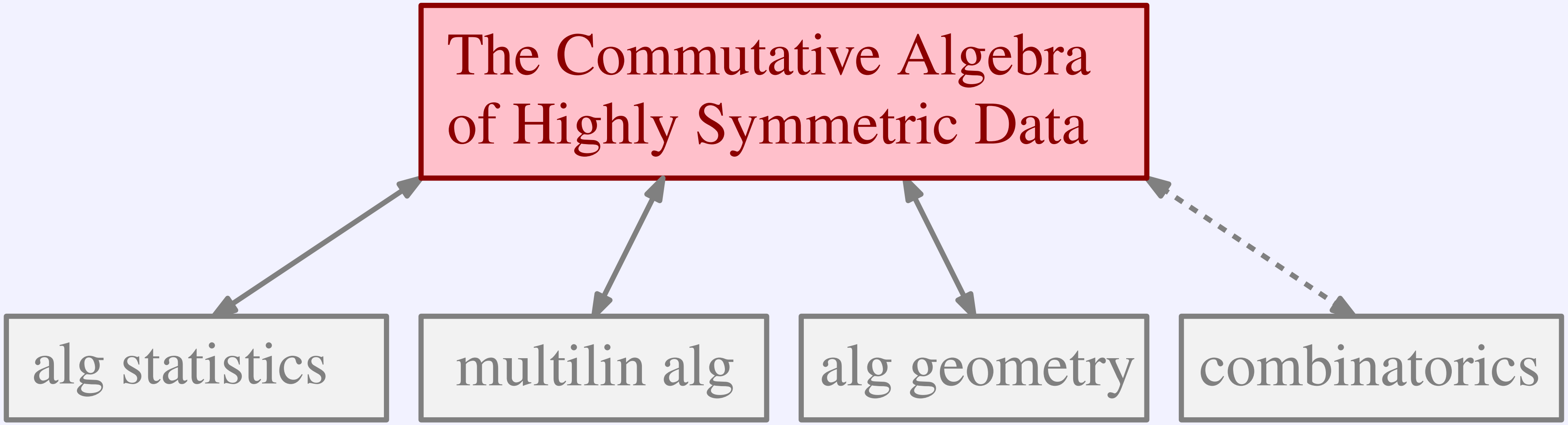


Correspondence
Equivalant to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N})$-Noetherianity of $\mathbf{Gr}_\infty(K)$ (but Noetherianity may be true even for infinite K).



⇝ *theory and algorithms for highly symmetric, ∞-dim varieties*

⇝ *exciting interplay of algebra, combinatorics, statistics, and geometry*



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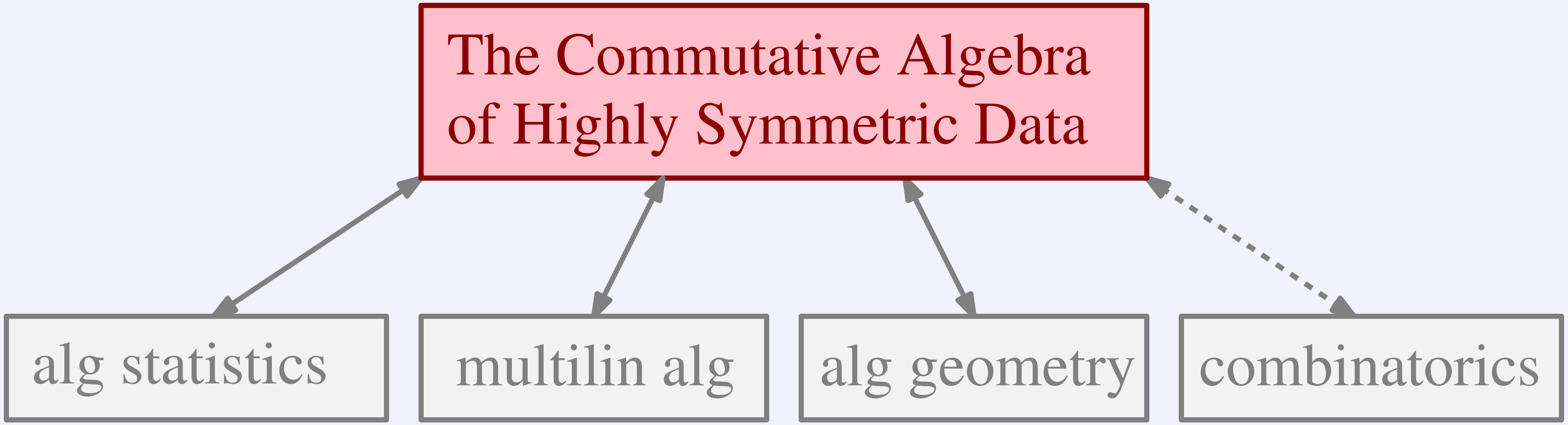

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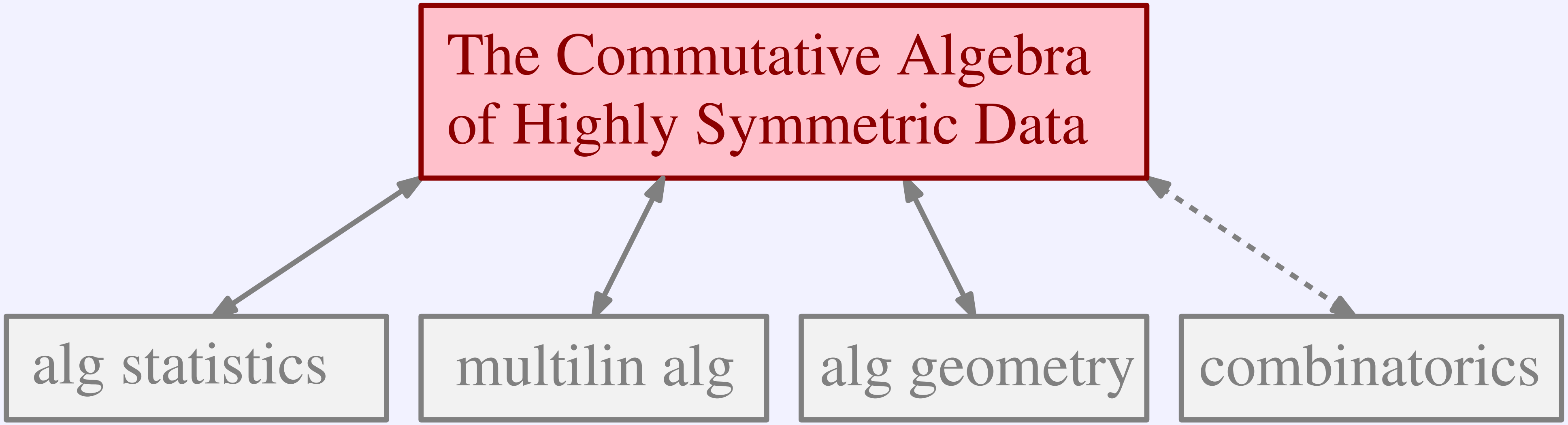
Ich habe mich davon überzeugt, daß
auch die Theologie ihre Vorzüge hat.



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Thank you!