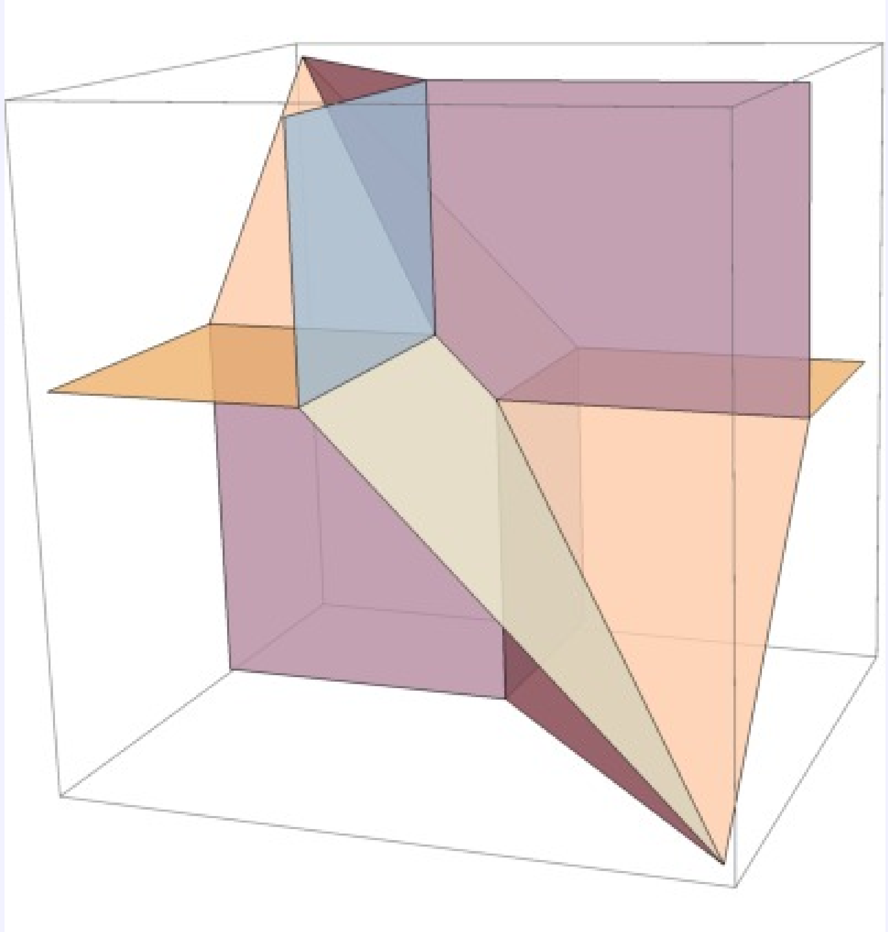
Algebraic matroids and Frobenius flocks



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Flocks

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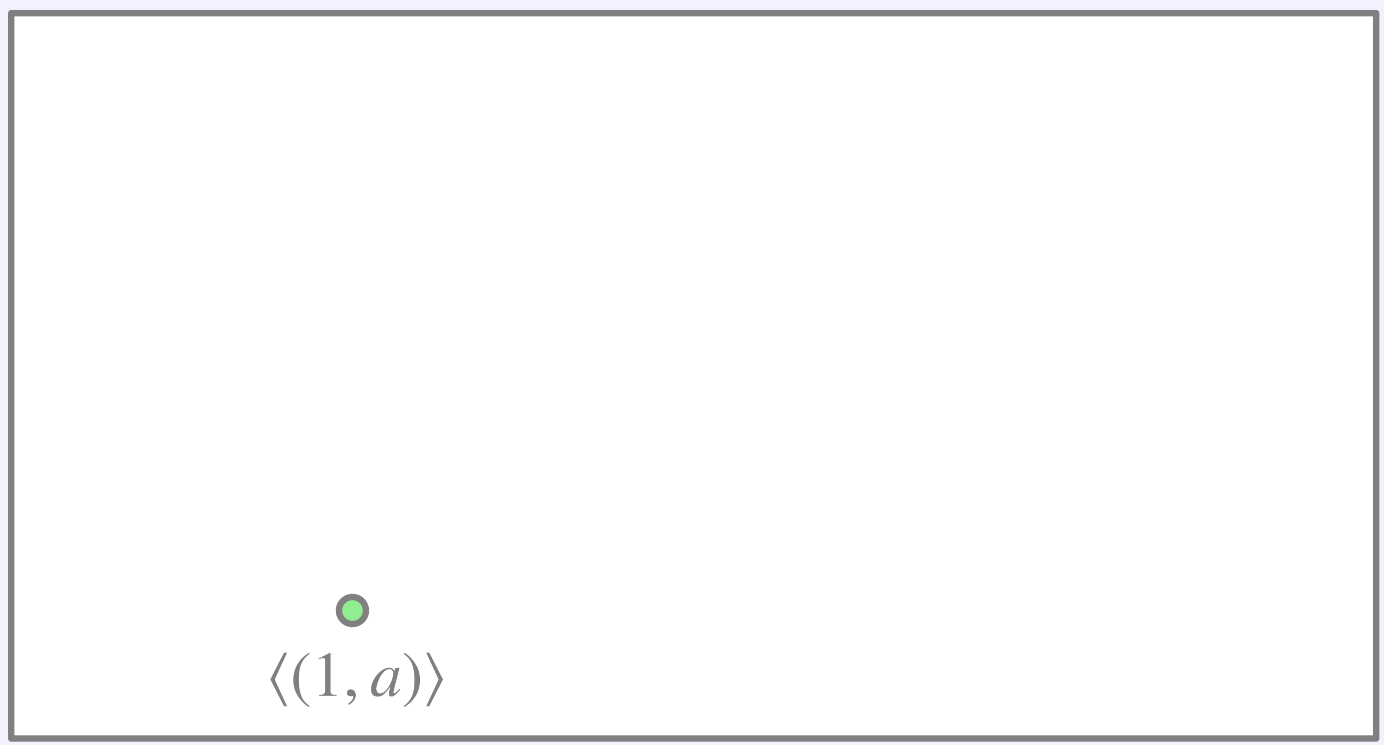
Definition
$V : \mathbb{Z}^E \to \{d\text{-dimensional subspaces of } K^E\}, \alpha \mapsto V_\alpha$
is a *vector space flock* on E over (K, σ) of rank d if
(VF1) $V_\alpha / i = V_{\alpha + e_i} \backslash i$ and
(VF2) $V_{\alpha + \mathbf{1}} = \mathbf{1} V_\alpha$.

$W \backslash i :=$ image of W in K^{E-i}
$W / i :=$ image of $e_i^\perp \cap W$ in K^{E-i}
$\mathbf{1}$ = the all-one vector

Example

(VF1) $V_\alpha / i = V_{\alpha + e_i} \backslash i$ (VF2) $V_{\alpha+\mathbf{1}} = \mathbf{1} V_\alpha$

$E = \{1, 2\}$, $V_0 = \langle (1, a) \rangle$, $a \neq 0$



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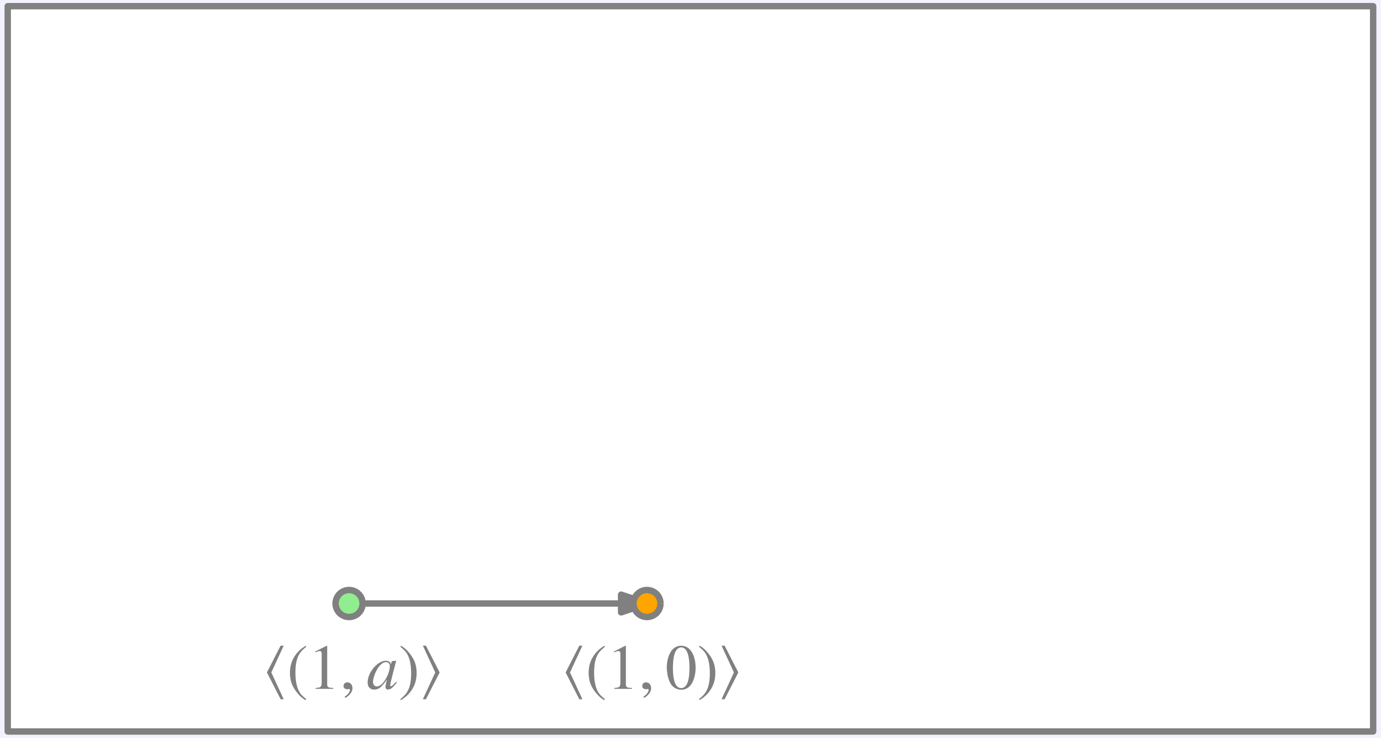

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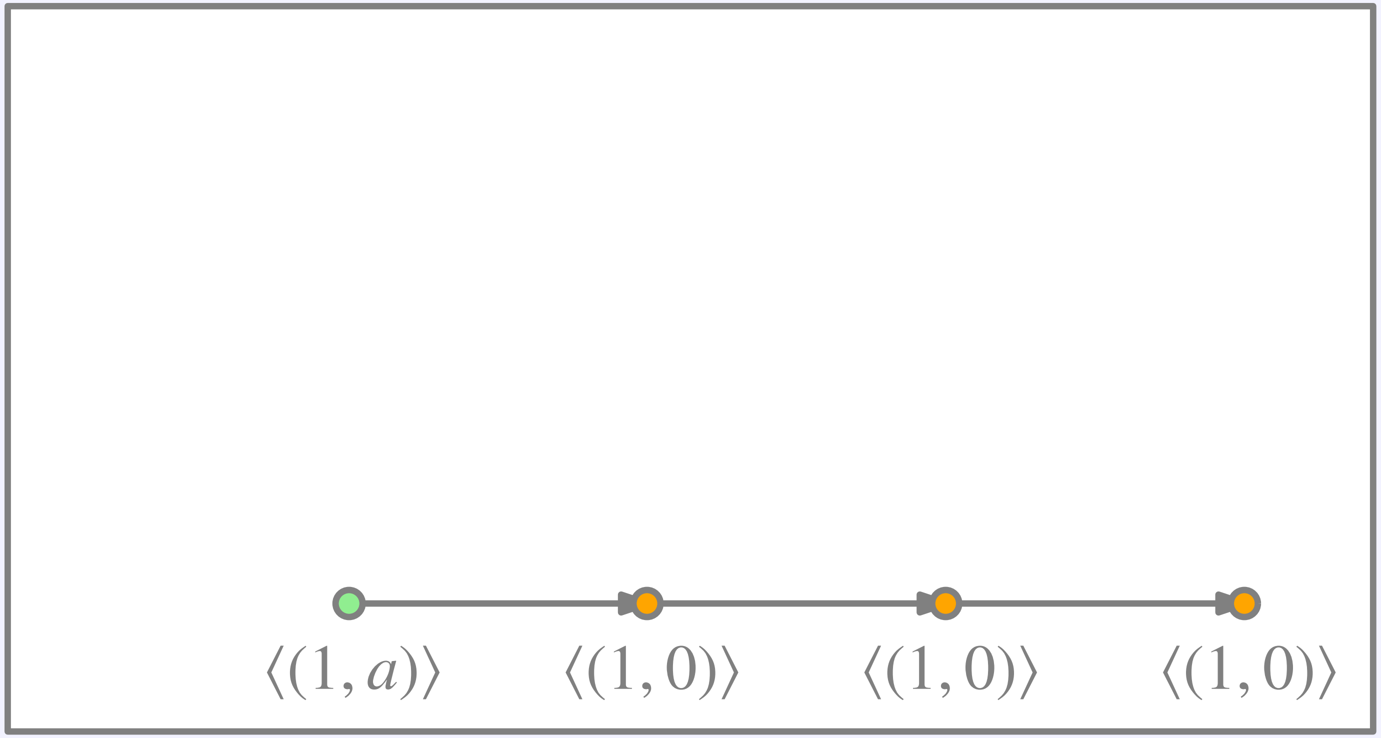

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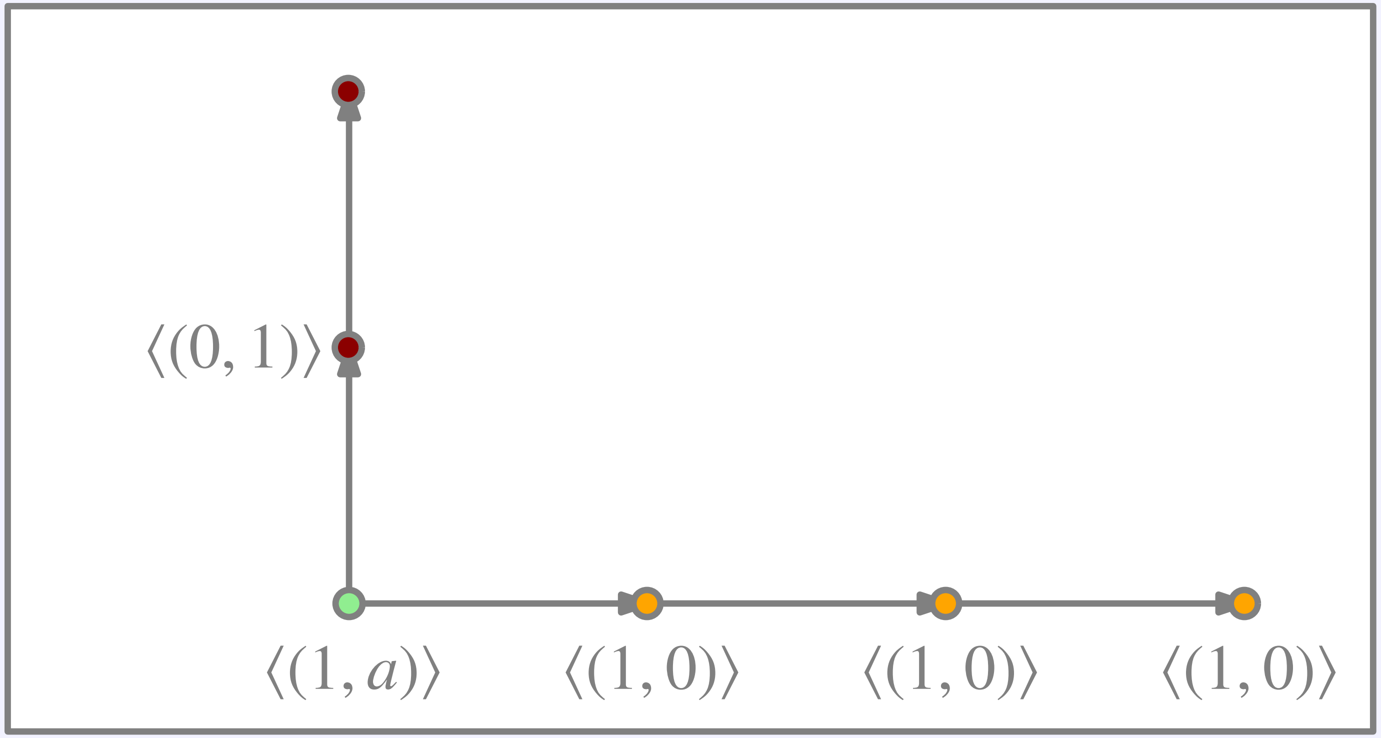
similarly $V_{ke_2} = \langle (0, 1) \rangle$ for $k \geq 1$



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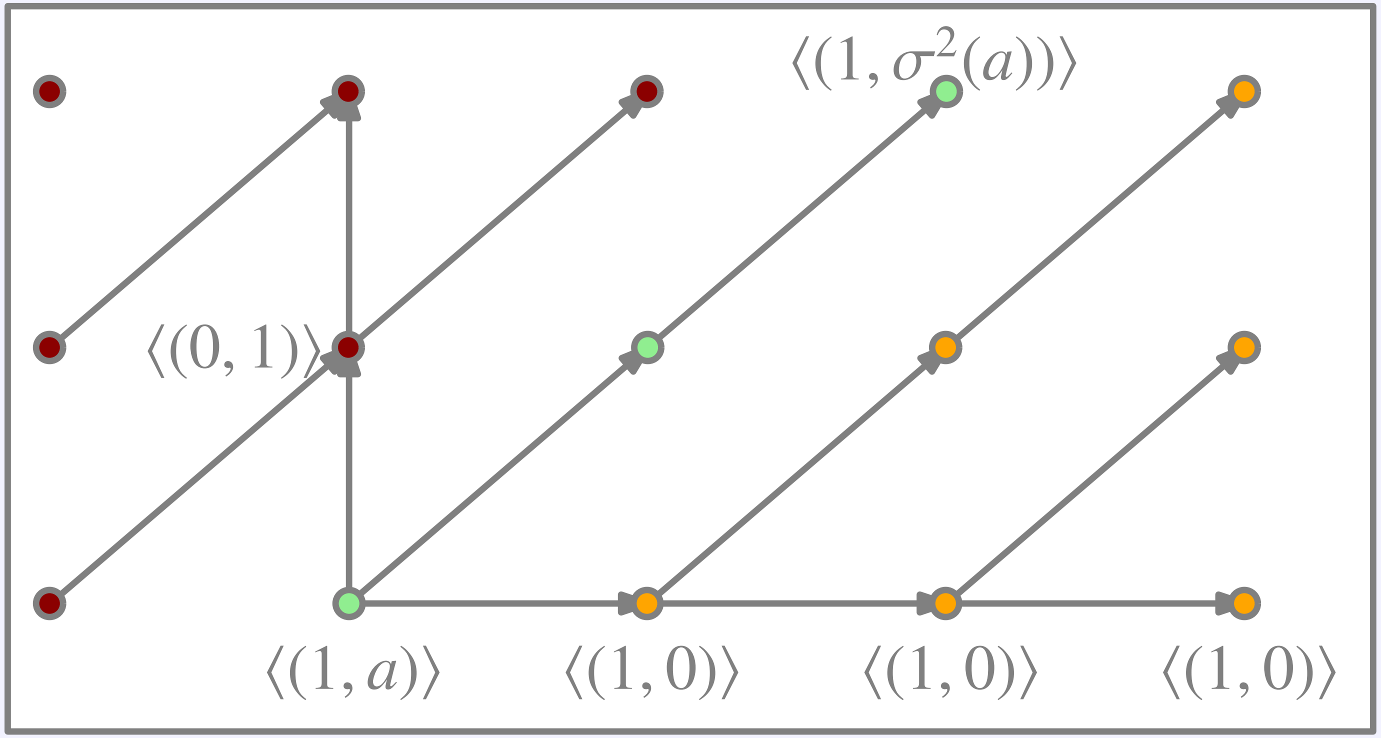
VF2 yields:



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But, (unfortunately?) many more.

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If V is a flock over (K, σ), then $V^* : \alpha \mapsto V^{\perp}_{-\alpha}$ is a flock over (K, σ^{-1}), and it satisfies $M(V^*) = M(V)^*$.

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- i is a coloop in $M(V_{\alpha+ke_i})$ for $k \gg 0$ unless it is a loop in $M(V)$;
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Minors

$V/i : \alpha \in \mathbb{Z}^{E-i} \to V_\beta/i$ for $\beta|_{E-i} = \alpha$ and $\beta_i \gg 0$; and
$V\backslash i : \alpha \in \mathbb{Z}^{E-i} \to V_\beta\backslash i$ for $\beta|_{E-i} = \alpha$ and $\beta_i \ll 0$
are flocks on $E - i$, satisfying
$M(V/i) = M(V)/i, M(V\backslash i) = M(V)\backslash i$, and $V^*/i = (V\backslash i)^*$.

Flocks from non-Archimedean fields

$\sigma : K \to K$ trivial, $W \subseteq K((t))^E$ a linear subspace
for $\alpha \in \mathbb{Z}^E$ set $t^\alpha W := \{(t^{\alpha_i} w_i)_i \mid w \in W\} \subseteq K((t))^E$

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Construction
$V : \alpha \mapsto V_\alpha :=$ image in K^E of $t^{-\alpha} W \cap K[[t]]^E$
is a vector space flock of rank $\dim W$ over $(K, 1_K)$.

$M(V_\alpha)$ has no loops if and only if $\alpha \in \mathrm{Trop}(W)$.

This suggests a tropical connection.

Algebraic matroids

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Ingleton's observation

If $\mathrm{char} K = 0$, then $M(X) = M(T_v X)$ for $v \in X$ general, so
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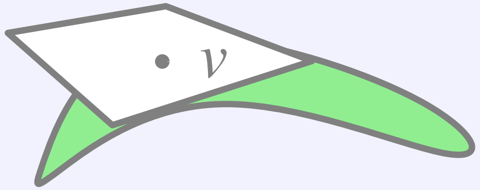

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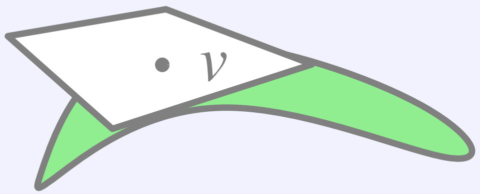
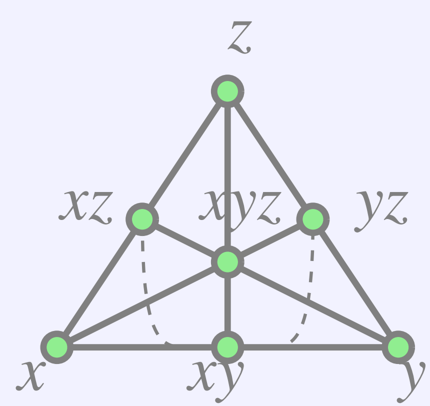


Example

This is *not* true in char, say, 2:

$X = \{(x, y, z, xy, xz, yz, xzy) \mid x, y, z \in \overline{\mathbb{F}_2}^3\}$

represents the *non-Fano* matroid, which is not linearly representable over $\overline{\mathbb{F}_2}$.



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Conditions (*)
v should satisfy the the following conditions *for all* $\alpha \in \mathbb{Z}^E$:
- αX is smooth at αv, and
- $M(T_{\alpha v}\alpha X) = M(T_{\xi_\alpha}\alpha X)$, where ξ is the generic point of αX.

To reduce from *very general* to just *general* we establish finiteness properties of flocks.

Example

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$$(-e_2 - e_3)X = \{(x, y, x^p + y, x + y^{(p^{g-1})}) \mid (x, y) \in K^2\}$$

$$T_0(-e_2 - e_3)X = \text{row space of} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; \text{ also } 2, 3 \text{ parallel.}$$

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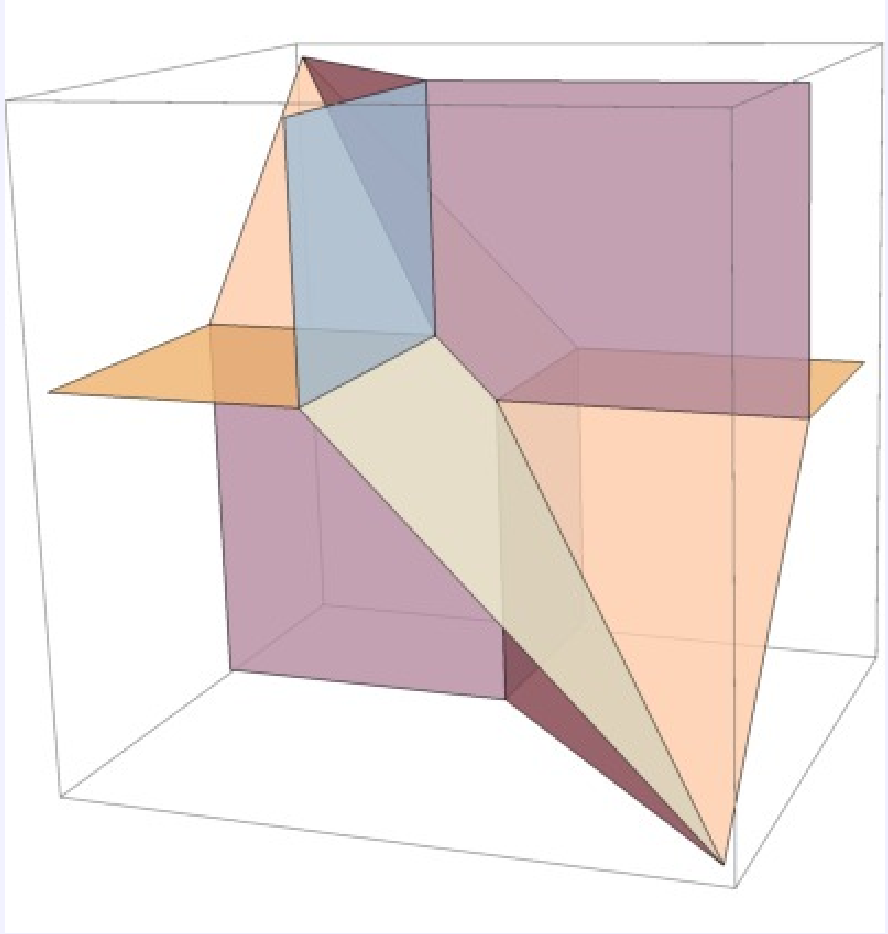
Example, continued

Cells where $M(T_0(\alpha X)) = M(T_{\xi_\alpha}\alpha X)$ is constant:



These cells are *alcoved polytopes*: max-plus and min-plus closed.

Matroid flocks from valuations and vice versa

Definition (Dress-Wenzel)

A *matroid valuation* is a map $\nu : \{d\text{-sets in } E\} \to \mathbb{R} \cup \{\infty\}$ such that $\nu(B) \neq \infty$ for some B and $\forall B, B',\ i \in B \setminus B'\ \exists j \in B' \setminus B :$ $\nu(B) + \nu(B') \geq \nu(B - i + j) + \nu(B' + i - j)$.

(ν then lies in the *Dressian*)

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Observations
$\nu \rightsquigarrow$ two matroids: $M^{\nu} := \{B \mid \nu(B) < \infty\}$ and $\{B \mid \nu(B) \text{ minimal}\}$; and $\nu'(B) := \nu(B) - \alpha \cdot e_B$ is a valuation for each $\alpha \in \mathbb{R}^E$.

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Theorem
Given a $\mathbb{Z} \cup \{\infty\}$-valued ν, set $M^\nu_\alpha := \{B \mid \nu(B) - \alpha \cdot e_B \text{ minimal}\}$ for each $\alpha \in \mathbb{Z}^E$. This satisfies matroid analogues of VF1,VF2. Conversely, each such *matroid flock* arises in this manner.

Overview

$\{$algebraic varieties $X \subseteq K^E\}$

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Murota–*thanks to Yu!*

{$\mathbb{Z} \cup \{\infty\}$-valued matroid valuations}

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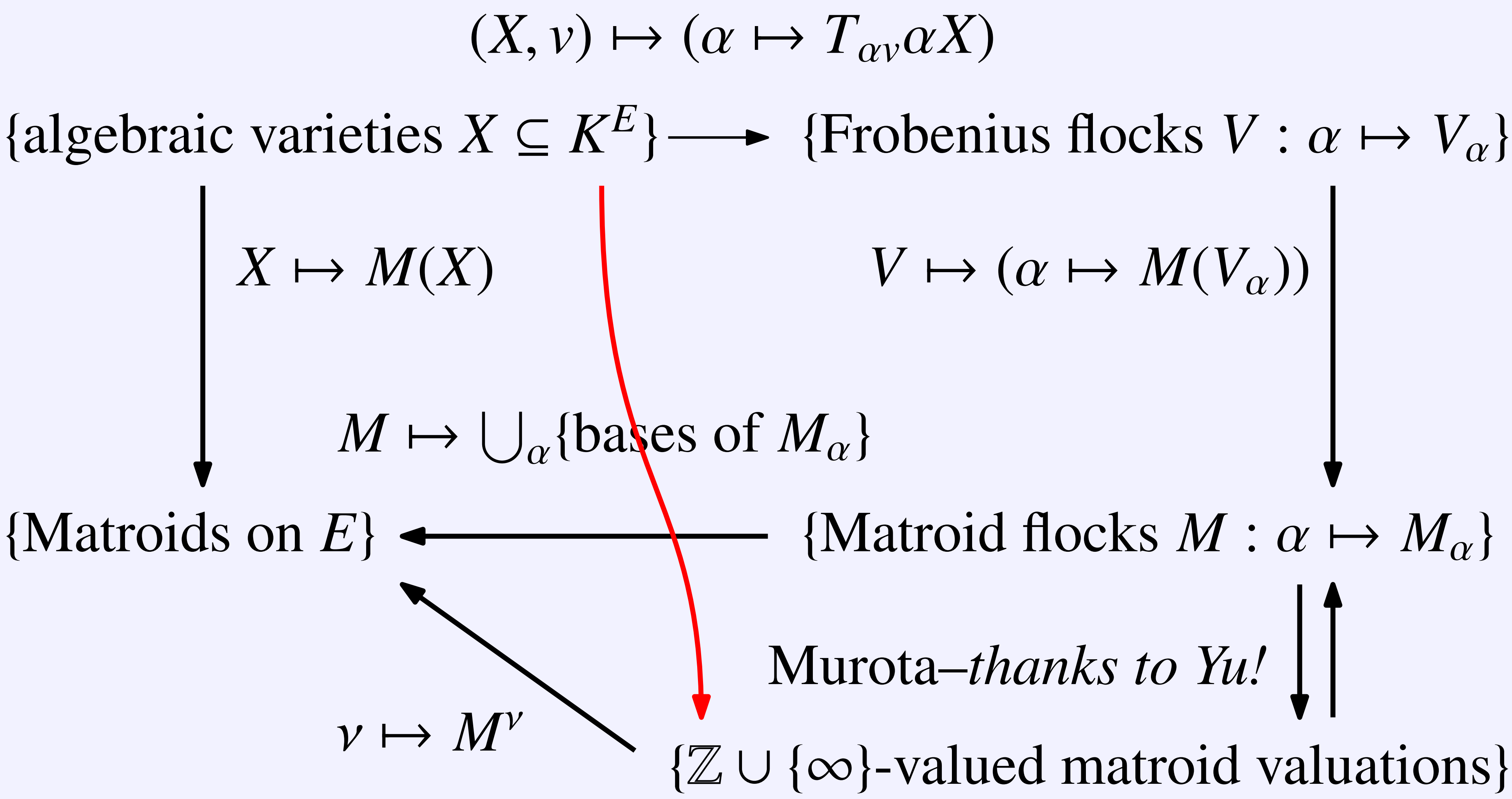

So to a d-dimensional algebraic variety $X \subseteq K^E$ in char p we associate the *Lindstrom valuation* $v^X : \{d\text{-subsets of } E\} \to \mathbb{Z} \cup \{\infty\}$.
Cartwright found a direct construction of v^X.

Cell decomposition

Proposition
ν a $\mathbb{Z} \cup \{\infty\}$-valued valuation and $\alpha, \beta \in \mathbb{R}^E$. Then $M^\nu_\alpha \supseteq M^\nu_\beta$ iff $\forall i \neq j \, \forall B \in M^\nu_\beta : \alpha_i - \alpha_j \geq \nu(B) - \nu(B - i + j)$.

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Consequences:
• A vector space flock can be specified by a finite amount of data.
• Conditions (*) on v are satisfied by general v: if αv satisfies it for αX, and if $M^{\nu_X}_\alpha = M^{\nu_X}_{\alpha+e_J}$, then $(e_J + \alpha)v$ satisfies it at $(\alpha + e_J)X$.

Rigidity

Definition (Dress-Wenzel)
A matroid M is *rigid* if every valuation ν with $M^\nu = M$ is of the form $M \to \mathbb{R},\ B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{R}^E$.

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Theorem
A rigid matroid is algebraically representable over an algebraically closed field K of positive characteristic if and only if it is linearly representable over K.

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Proof
If X is an algebraic representation, then the Lindström valuation $v^X : M(X) \to \mathbb{Z}$ sends $B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{R}^E$. Then $M^v_\alpha = M^v$, and the cell where this happens also contains an integral α. Now $M(X) = M(T_{\alpha v}\alpha X)$ for $v \in X$ general. □

(Non-)examples

Using this theorem, we can re-prove several existing results, such as: the projective plane over $\mathbb{F}_p$ is algebraically representable only over fields of characteristic p (Lindström).

To extend the applicability of this method, we are trying to relax the condition that the matroid be rigid.

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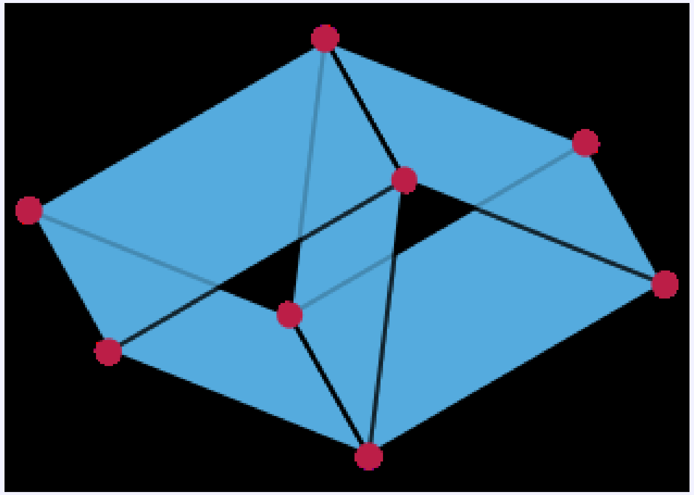
To extend the applicability of this method, we are trying to relax the condition that the matroid be rigid.

However, using Frobenius flocks alone one cannot reprove all non-algebraicity results:

Theorem-in-progress
If M has a flock representation over (K,σ), and H is a circuit hyperplane in M, then the matroid obtained by turning H into a basis again has a flock representation over (K,σ).

The missing diagonal plane is independent:



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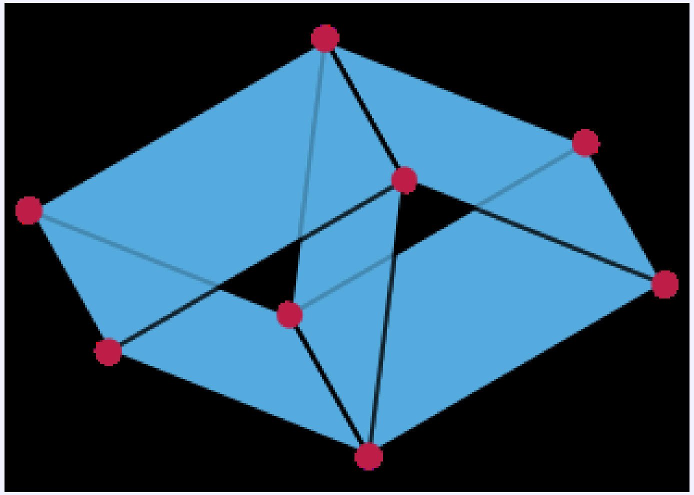

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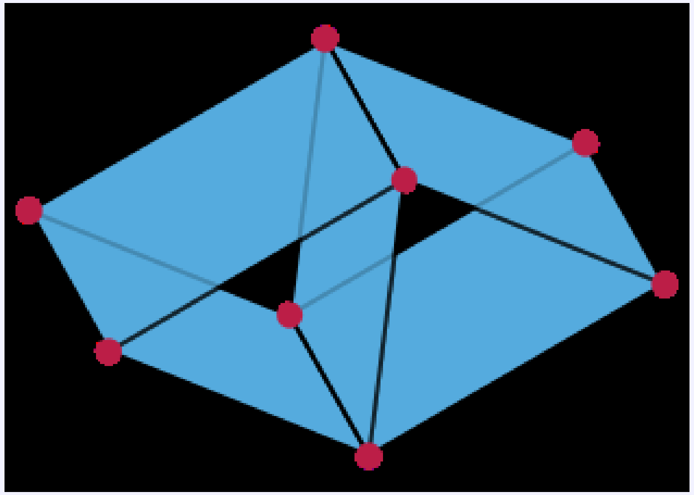

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