Stabilisation in algebra and geometry

Jan Draisma
Universität Bern

Fribourg colloquium, March 2017

Central question

Given an infinite collection $(X_n)_n$ *of algebro-geometric structures, are they characterised by finitely many among them?*

Central question

Given an infinite collection $(X_n)_n$ *of algebro-geometric structures, are they characterised by finitely many among them?*

n runs through $\mathbb{N}$, or through all finite sets or trees, or …

have maps $X_n \to X_m$ that propagate structure

Central question

Given an infinite collection $(X_n)_n$ *of algebro-geometric structures, are they characterised by finitely many among them?*

n runs through $\mathbb{N}$, or through all finite sets or trees, or ...

have maps $X_n \to X_m$ that propagate structure

Topic 1 (Gaussian two-factor model)
$X_n := \{SS^T + D \mid S \in \mathbb{R}^{n\times 2}, D \text{ diag} > 0\}$
any $\{i_1 < \ldots < i_m\} \subseteq \{1, \ldots, n\}$ gives a map $X_n \to X_m$

Central question

Given an infinite collection $(X_n)_n$ of algebro-geometric structures, are they characterised by finitely many among them?

n runs through $\mathbb{N}$, or through all finite sets or trees, or …

have maps $X_n \to X_m$ that propagate structure

Topic 1 (Gaussian two-factor model)
$X_n := \{SS^T + D \mid S \in \mathbb{R}^{n\times 2}, D \text{ diag} > 0\}$
any $\{i_1 < \ldots < i_m\} \subseteq \{1, \ldots, n\}$ gives a map $X_n \to X_m$

Theorem [Drton-Xiao, 2010]
$\Sigma \in \mathbb{R}^{n\times n}$, PD, is in X_n iff all 6×6 principal submatrices are in X_6.

X_n is given by polynomial eqs and ineqs; we will focus on the eqs.

(Non-)Noetherianity of rings

Theorem [Hilbert, 1890]

For a field K, any ideal in $K[x_1, \dots, x_n]$ is finitely generated.

uses *Dickson's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^n_{\geq 0} \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$

(Non-)Noetherianity of rings

Theorem [Hilbert, 1890]

For a field K, any ideal in $K[x_1, \ldots, x_n]$ is finitely generated.

uses *Dickson's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^n_{\geq 0} \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$

Varying the number of variables

For every finite set S set $R_S := K[x_i \mid i \in S]$, and for injective $\sigma : S \to T$ consider $\sigma : R_S \to R_T, x_i \mapsto x_{\sigma(i)}$.

(Non-)Noetherianity of rings

Theorem [Hilbert, 1890]
For a field K, any ideal in $K[x_1, \ldots, x_n]$ is finitely generated.

uses *Dickson's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^n_{\geq 0} \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$

Varying the number of variables
For every finite set S set $R_S := K[x_i \mid i \in S]$, and for injective $\sigma : S \to T$ consider $\sigma : R_S \to R_T, x_i \mapsto x_{\sigma(i)}$.

Theorem [Cohen, 1967; Aschenbrenner-Hillar, 2007]
For every finite set S, let I_S be an ideal in R_S such that $\sigma : S \to T$ maps I_S into I_T. Then $I_\bullet$ is generated by $I_\emptyset, \ldots, I_{[n_0]}$ for some n_0.

uses *Higman's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^*_{\geq 0} \Rightarrow \exists i < j : \alpha_i \leq \alpha_j$

(Non-)Noetherianity of rings

Theorem [Hilbert, 1890]
For a field K, any ideal in $K[x_1, \ldots, x_n]$ is finitely generated.

uses *Dickson's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^n_{\geq 0} \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$

Varying the number of variables
For every finite set S set $R_S := K[x_i \mid i \in S]$, and for injective $\sigma : S \to T$ consider $\sigma : R_S \to R_T, x_i \mapsto x_{\sigma(i)}$.

Theorem [Cohen, 1967; Aschenbrenner-Hillar, 2007]
For every finite set S, let I_S be an ideal in R_S such that $\sigma : S \to T$ maps I_S into I_T. Then $I_\bullet$ is generated by $I_\emptyset, \ldots, I_{[n_0]}$ for some n_0.

uses *Higman's Lemma*: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^*_{\geq 0} \Rightarrow \exists i < j : \alpha_i \leq \alpha_j$

same thm for $K[x_{ij} | i \in S, j \in [k]]$ but *not* for $K[x_{ij} \mid i, j \in S]$

Topic 1, continued [Drton-Sturmfels-Sullivant, 2007]

$X_n \subseteq \mathbb{R}^{n\times n}$ 2-factor model, vanishing ideal $I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]]$

Topic 1, continued [Drton-Sturmfels-Sullivant, 2007]

$X_n \subseteq \mathbb{R}^{n\times n}$ 2-factor model, vanishing ideal $I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]]$

$x_{ij} - x_{ji} \in I_n$ for $n \geq 2$

off-diagonal 3×3-subdeterminants $\in I_n$ for $n \geq 6$

$\sum_{\pi\in\mathrm{Sym}(5)} \mathrm{sgn}(\pi)\pi \cdot x_{12}x_{23}x_{34}x_{45}x_{51} \in I_5 \rightsquigarrow$ eqs for $n \geq 5$

Topic 1, continued [Drton-Sturmfels-Sullivant, 2007]

$X_n \subseteq \mathbb{R}^{n\times n}$ 2-factor model, vanishing ideal $I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]]$

$x_{ij} - x_{ji} \in I_n$ for $n \geq 2$

off-diagonal 3×3-subdeterminants $\in I_n$ for $n \geq 6$

$\sum_{\pi\in \mathrm{Sym}(5)} \mathrm{sgn}(\pi)\pi \cdot x_{12}x_{23}x_{34}x_{45}x_{51} \in I_5 \rightsquigarrow$ eqs for $n \geq 5$

Theorem [Brouwer-D, 2011]

These generate I_n for all $n \geq 6$ via injections $[6] \to [n]$.

Topic 1, continued [Drton-Sturmfels-Sullivant, 2007]

$X_n \subseteq \mathbb{R}^{n\times n}$ 2-factor model, vanishing ideal $I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]]$

$x_{ij} - x_{ji} \in I_n$ for $n \geq 2$
off-diagonal 3×3-subdeterminants $\in I_n$ for $n \geq 6$
$\sum_{\pi\in \mathrm{Sym}(5)} \mathrm{sgn}(\pi)\pi \cdot x_{12}x_{23}x_{34}x_{45}x_{51} \in I_5 \rightsquigarrow$ eqs for $n \geq 5$

Theorem [Brouwer-D, 2011]

These generate I_n for all $n \geq 6$ via injections $[6] \to [n]$.

Replacing 2 *by k we know only weaker stabilisation:*

Theorem [D, 2010]

$\forall k\ \exists n_0$ such that via injections $[n_0] \to [n]$ the ideal I_{n_0} generates I_n *up to radical.*

Instances of stabilisation

(using *Noetherianity up to symmetry*)

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

Theorem [D-Kuttler, 2014]

For any fixed k there is a d, *independent of n and the V_i*, such that $\overline{\{T \text{ of rank} \leq k\}}$ is defined by polynomials of degree $\leq d$.

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

Theorem [D-Kuttler, 2014]

For any fixed k there is a d, *independent of n and the V_i*, such that $\overline{\{T \text{ of rank} \leq k\}}$ is defined by polynomials of degree $\leq d$.

Table

k	0	1	2	3	4
d	1	2	$3^{\dagger}$	$4^{\bullet}$	$\geq 9^{*}$

† [Landsberg-Manivel, 2004]
• [Qi, 2014]
∗ [Strassen,1983]

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

Theorem [D-Kuttler, 2014]

For any fixed k there is a d, *independent of n and the V_i*, such that $\overline{\{T \text{ of rank} \leq k\}}$ is defined by polynomials of degree $\leq d$.

Table

k	0	1	2	3	4
d	1	2	$3^{\dagger}$	$4^{\bullet}$	$\geq 9^{*}$

† [Landsberg-Manivel, 2004]
• [Qi, 2014]
∗ [Strassen,1983]

relevant maps from $X(V_1, \ldots, V_n) = \overline{\{\text{rank} \leq k\}} \subseteq V_1 \otimes \cdots \otimes V_k$ into $X(W_1, \ldots, W_n)$ or $X(V_1, \ldots, V_{n-1} \otimes V_n)$ or $X(V_{\pi(1)}, \ldots, V_{\pi(n)})$

Definition

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of T as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$.

Theorem [D-Kuttler, 2014]

For any fixed k there is a d, *independent of n and the V_i*, such that $\overline{\{T \text{ of rank} \leq k\}}$ is defined by polynomials of degree $\leq d$.

Table

k	0	1	2	3	4
d	1	2	$3^{\dagger}$	$4^{\bullet}$	$\geq 9^{*}$

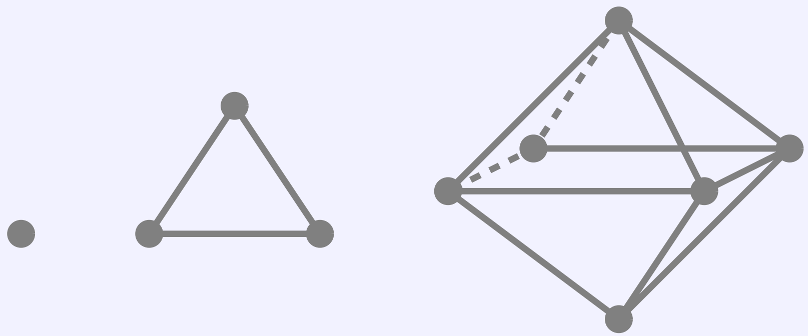
† [Landsberg-Manivel, 2004]
• [Qi, 2014]
∗ [Strassen,1983]

relevant maps from $X(V_1, \ldots, V_n) = \overline{\{\text{rank} \leq k\}} \subseteq V_1 \otimes \cdots \otimes V_k$ into $X(W_1, \ldots, W_n)$ or $X(V_1, \ldots, V_{n-1} \otimes V_n)$ or $X(V_{\pi(1)}, \ldots, V_{\pi(n)})$

Snowden has a stabilisation result for higher syzygies for $k = 1$.

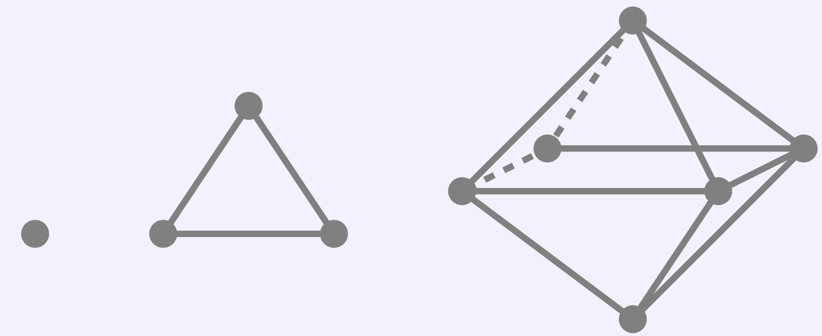
Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



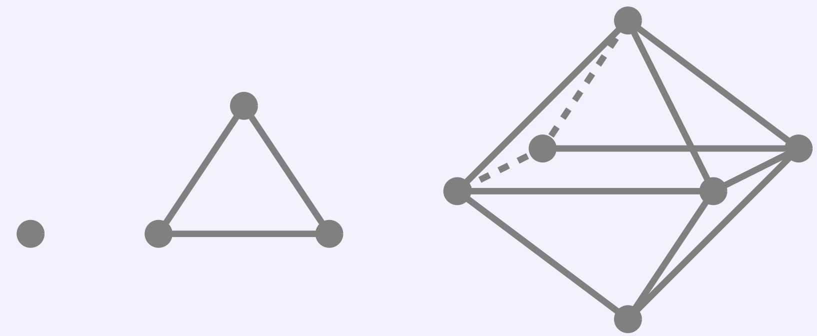
Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



Theorem [De Loera-Sturmfels-Thomas, 1995]

P_n has a Markov basis consisting of *moves* $v_{ij} + v_{kl} \rightarrow v_{il} + v_{kj}$ and $v_{ij} \rightarrow v_{ji}$ for i, j, k, l distinct; i.e., if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$, then the expressions are connected by such moves.



Second hypersimplex

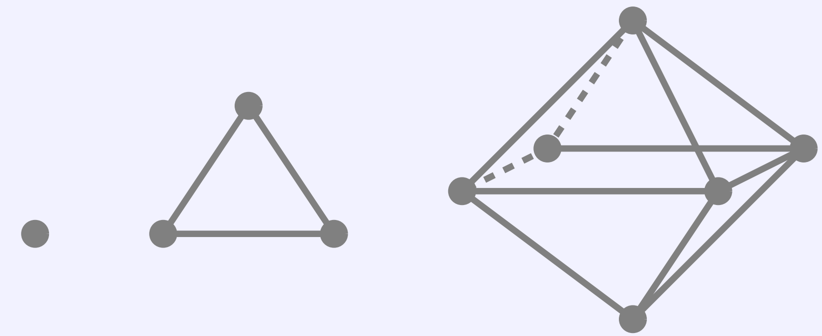
$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$

Theorem [De Loera-Sturmfels-Thomas, 1995]

P_n has a Markov basis consisting of *moves* $v_{ij} + v_{kl} \to v_{il} + v_{kj}$ and $v_{ij} \to v_{ji}$ for i, j, k, l distinct; i.e., if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$, then the expressions are connected by such moves.

Theorem [D-Eggermont-Krone-Leykin 2016]

Any sequence $(P_n \subseteq \mathbb{Z}^n)_n$ of lattice point configurations such that $P_n = \mathrm{Sym}(n) P_{n-1}$ for $n \gg 0$ admits a sequence $(M_n)_n$ of Markov bases such that $M_n = \mathrm{Sym}(n) M_{n-1}$ for $n \gg 0$.



Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$

Theorem [De Loera-Sturmfels-Thomas, 1995]

P_n has a Markov basis consisting of *moves* $v_{ij} + v_{kl} \to v_{il} + v_{kj}$ and $v_{ij} \to v_{ji}$ for i, j, k, l distinct; i.e., if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$, then the expressions are connected by such moves.

Theorem [D-Eggermont-Krone-Leykin 2016]

Any sequence $(P_n \subseteq \mathbb{Z}^n)_n$ of lattice point configurations such that $P_n = \mathrm{Sym}(n) P_{n-1}$ for $n \gg 0$ admits a sequence $(M_n)_n$ of Markov bases such that $M_n = \mathrm{Sym}(n) M_{n-1}$ for $n \gg 0$.

(Also true for $P_n \subseteq \mathbb{Z}^{k \times n}$, considered a subset of $\mathbb{Z}^{k \times (n+1)}$ by adding a zero column. We also have an algorithm for computing $(M_n)_n$.)

Topic 4: homological stability

M a compact manifold
for a finite set S define $C_S(M) := \{(p_i)_{i \in S} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq M^S$
for any injection $S \subseteq T$ have map $C_T(M) \to C_S(M)$
dually: $H^d(C_S(M), \mathbb{Q}) \to H^d(C_T(M), \mathbb{Q})$.

Topic 4: homological stability

M a compact manifold
for a finite set S define $C_S(M) := \{(p_i)_{i\in S} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq M^S$
for any injection $S \subseteq T$ have map $C_T(M) \to C_S(M)$
dually: $H^d(C_S(M), \mathbb{Q}) \to H^d(C_T(M), \mathbb{Q})$.

Theorem [Church, 2012]
For a fixed d and M, finitely many of these cohomology groups generate the other ones via these maps.

Topic 4: homological stability

M a compact manifold
for a finite set S define $C_S(M) := \{(p_i)_{i \in S} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq M^S$
for any injection $S \subseteq T$ have map $C_T(M) \to C_S(M)$
dually: $H^d(C_S(M), \mathbb{Q}) \to H^d(C_T(M), \mathbb{Q})$.

Theorem [Church, 2012]
For a fixed d and M, finitely many of these cohomology groups generate the other ones via these maps.

Among other things, this implies that the $\mathrm{Sym}(S)$*-character of* $H^d(C_S(M), \mathbb{Q})$ *is constant for* $|S| \gg 0$.

The map $S \mapsto H^d(C_S(M), \mathbb{Q})$ is an example of an *FI-module*; their structure has been studied intensively by Church, Ellenberg, Farb.

Grassmannians

$\mathrm{Gr}_k(V)$ is a variety parameterising k-dimensional subspaces of V. It is functorial in V, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$.

Grassmannians

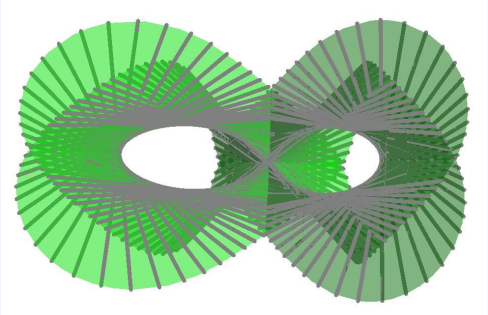
$\mathrm{Gr}_k(V)$ is a variety parameterising k-dimensional subspaces of V. It is functorial in V, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$.

A sequence $X_0, X_1, X_2, \ldots$ of rules $V \mapsto X_k(V) \subseteq \bigwedge^k(V)$ satisfying these two properties is called a *Plücker variety*.

Grassmannians

$\mathrm{Gr}_k(V)$ is a variety parameterising k-dimensional subspaces of V. It is functorial in V, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$.

A sequence $X_0, X_1, X_2, \ldots$ of rules $V \mapsto X_k(V) \subseteq \bigwedge^k(V)$ satisfying these two properties is called a *Plücker variety*.



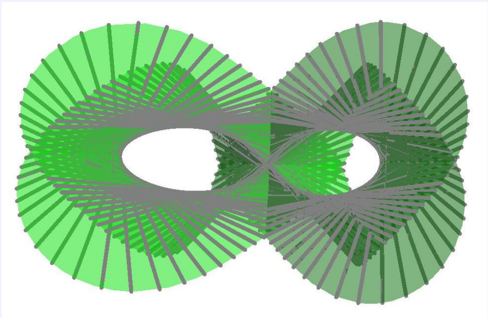
Construction of new Plücker varieties

tangential variety, secant variety, etc.

Grassmannians

$\mathrm{Gr}_k(V)$ is a variety parameterising k-dimensional subspaces of V. It is functorial in V, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$.

A sequence $X_0, X_1, X_2, \ldots$ of rules $V \mapsto X_k(V) \subseteq \bigwedge^k(V)$ satisfying these two properties is called a *Plücker variety*.



Construction of new Plücker varieties

tangential variety, secant variety, etc.

Theorem [D-Eggermont 2014]

For *bounded* Plücker varieties, $(X_k(K^n))_{k,n-k}$ stabilises.

(*For* $X = \mathrm{Gr}$, $X_\infty =$ *Sato's Grassmannian* $\subseteq$ *dual infinite wedge.*)

Topic 6: Stillman's conjecture

$R = K[x_1, \ldots, x_N]$

$f_1, \ldots, f_k$ homogeneous polynomials of degrees $d_1, \ldots, d_k$ in R

$I := (f_1, \ldots, f_k)$ the ideal they generate

Topic 6: Stillman's conjecture

$R = K[x_1, \ldots, x_N]$
$f_1, \ldots, f_k$ homogeneous polynomials of degrees $d_1, \ldots, d_k$ in R
$I := (f_1, \ldots, f_k)$ the ideal they generate

Conjecture [Stillman]
Fixing $d_1, \ldots, d_k$ but not N or the f_i, algebraic invariants of R/I such as its projective dimension should be uniformly bounded.

Topic 6: Stillman's conjecture

$R = K[x_1, \ldots, x_N]$
$f_1, \ldots, f_k$ homogeneous polynomials of degrees $d_1, \ldots, d_k$ in R
$I := (f_1, \ldots, f_k)$ the ideal they generate

Conjecture [Stillman]
Fixing $d_1, \ldots, d_k$ but not N or the f_i, algebraic invariants of R/I such as its projective dimension should be uniformly bounded.

Theorem [Hochster-Ananyan 2016]
This is indeed true for projective dimension.

Topic 6: Stillman's conjecture

$R = K[x_1, \ldots, x_N]$
$f_1, \ldots, f_k$ homogeneous polynomials of degrees $d_1, \ldots, d_k$ in R
$I := (f_1, \ldots, f_k)$ the ideal they generate

Conjecture [Stillman]
Fixing $d_1, \ldots, d_k$ but not N or the f_i, algebraic invariants of R/I such as its projective dimension should be uniformly bounded.

Theorem [Hochster-Ananyan 2016]
This is indeed true for projective dimension.

Related question
Is the functor $V \mapsto S^{d_1}V \oplus \cdots \oplus S^{d_k}V$ topologically Noetherian?

Topic 6: Stillman's conjecture

$R = K[x_1, \ldots, x_N]$
$f_1, \ldots, f_k$ homogeneous polynomials of degrees $d_1, \ldots, d_k$ in R
$I := (f_1, \ldots, f_k)$ the ideal they generate

Conjecture [Stillman]
Fixing $d_1, \ldots, d_k$ but not N or the f_i, algebraic invariants of R/I such as its projective dimension should be uniformly bounded.

Theorem [Hochster-Ananyan 2016]
This is indeed true for projective dimension.

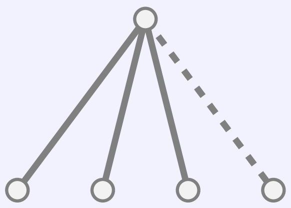
Related question
Is the functor $V \mapsto S^{d_1}V \oplus \cdots \oplus S^{d_k}V$ topologically Noetherian?

Theorem [Derksen-Eggermont-Snowden 2017]
Yes for $k = 1$ and $d_1 = 3$.

Stabilisation in other areas

Algebraic statistics

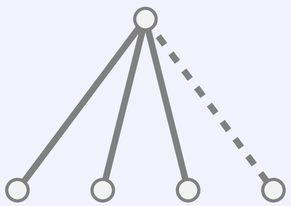
families of graphical models where the graph grows



[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont, D-Oosterhof,. . .]

Stabilisation in other areas

Algebraic statistics
families of graphical models where the graph grows
[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont, D-Oosterhof,. . .]



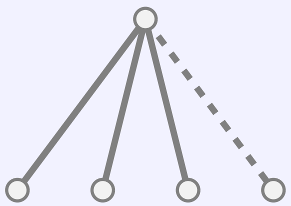
Commutative algebra and representation theory
higher syzygies, sequences of modules
[Sam-Snowden, Church-Ellenberg-Farb, D-Krone-Leykin]

Stabilisation in other areas

Algebraic statistics

families of graphical models where the graph grows

[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont, D-Oosterhof,. . .]



Commutative algebra and representation theory

higher syzygies, sequences of modules

[Sam-Snowden, Church-Ellenberg-Farb, D-Krone-Leykin]

Thank you.