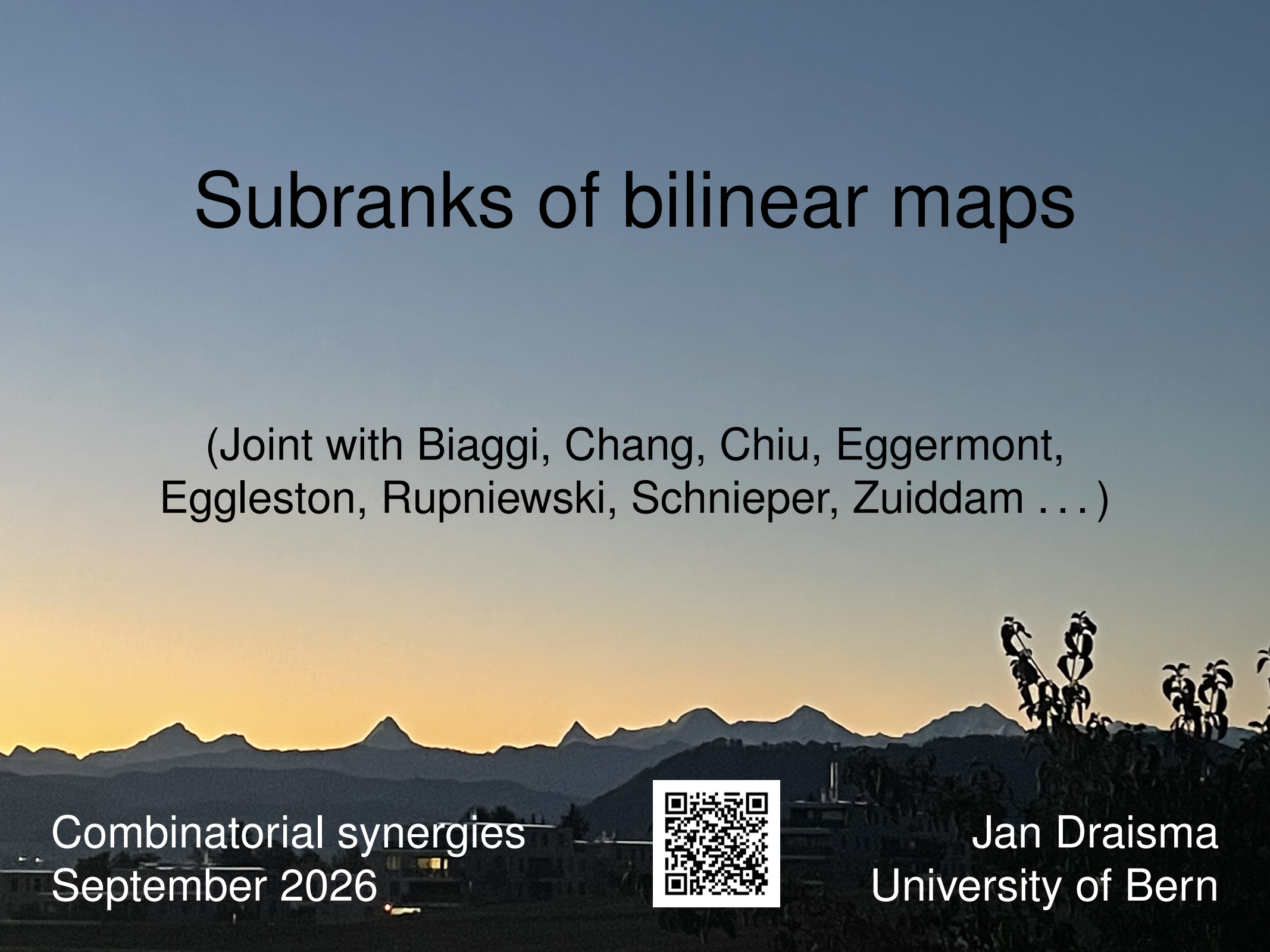
Subranks of bilinear maps

(Joint with Biaggi, Chang, Chiu, Eggermont, Eggleston, Rupniewski, Schnieper, Zuiddam . . .)



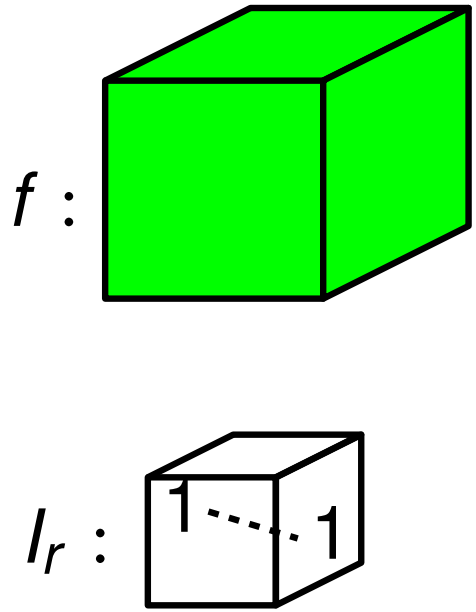
Combinatorial synergies
September 2026

Jan Draisma
University of Bern

Subrank

$$\begin{array}{ccc} U \times V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \alpha \times \beta}\uparrow & \circlearrowright & \downarrow{\scriptstyle \gamma} \\ K^r \times K^r & \longrightarrow & K^r \end{array}$$

$$I_r : (a, b) \mapsto a * b := (a_i b_i)_i$$



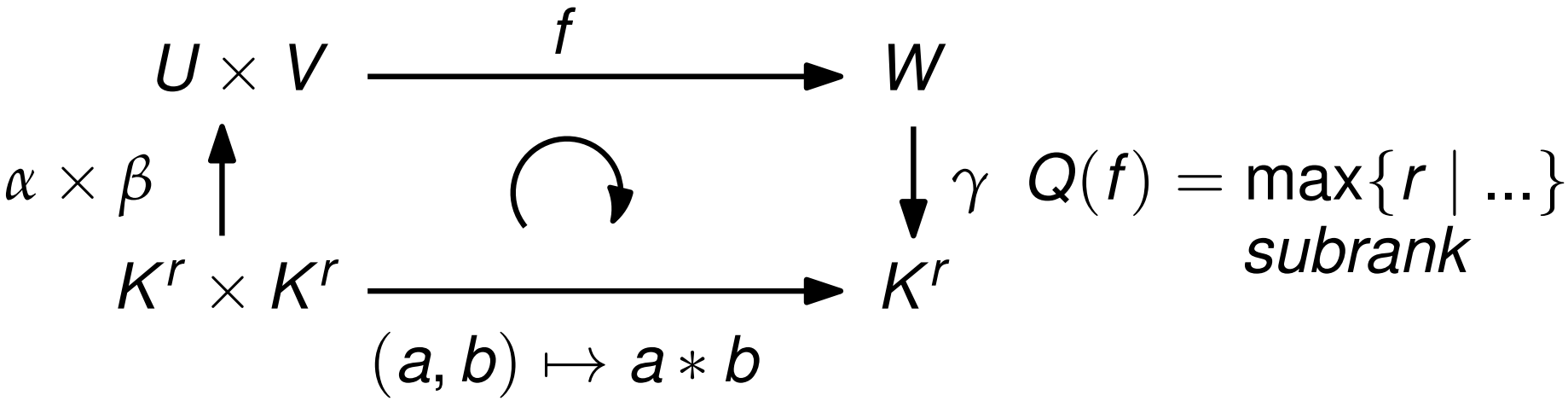
Definition [Strassen]

The *subrank* of f is $Q(f) := \max\{r \mid \exists \alpha, \beta, \gamma : \gamma \circ f \circ (\alpha \times \beta)$ is the Hadamard product $I_r : K^r \times K^r \to K^r\}$.

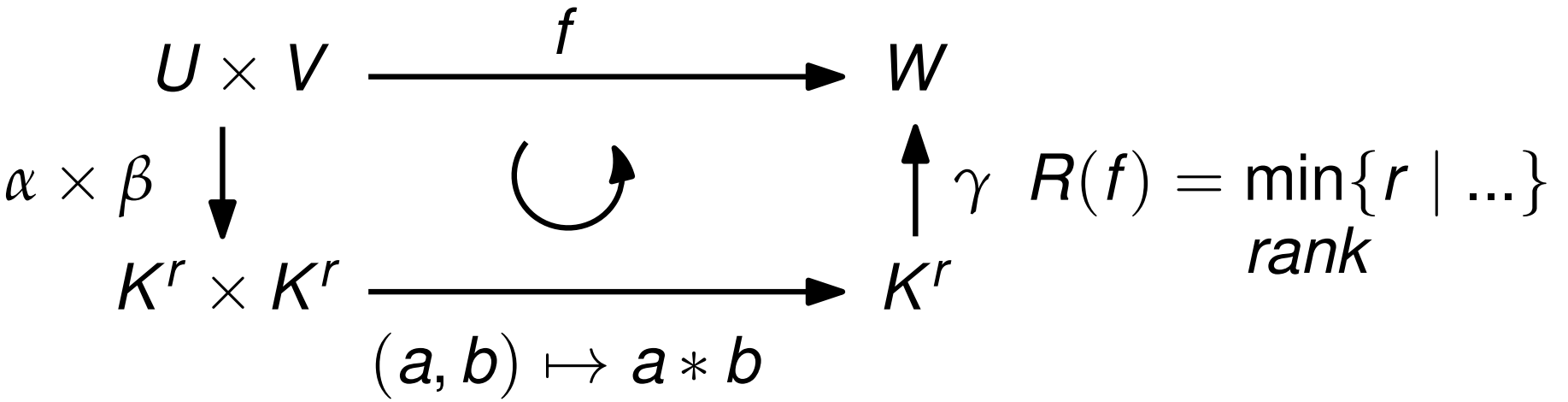
Remark. α and β are injective, γ is surjective, so $Q(f) \leq \min\{\dim(U), \dim(V), \dim(W)\}$.

Example. $Q(f) = 0 \Leftrightarrow f = 0$, and $Q(I_r) = r$.

Relation to ordinary tensor rank



$Q(f) = \max\{r \mid ...\}$
subrank



$R(f) = \min\{r \mid ...\}$
rank

Remark. If f *concise* (surjective, zero radical), then

$$Q(f) \leq \min\{\dim(U), \dim(V), \dim(W)\}$$
$$\leq \max\{\dim(U), \dim(V), \dim(W)\} \leq R(f).$$

A case study: $K[x]/(x^d)$

Alternative characterisation

$Q(f) = \max\{r \mid \exists u_1, \ldots, u_r \in U, v_1, \ldots v_r \in V : (f(u_i, v_i))_i$ is linearly independent modulo $(f(u_i, v_j))_{i \neq j}\}$.

Example

$U = V = W = K[x]/(x^d)$, $f =$multiplication.

Monomial subrank

Take $u_i = x^{a_i}$ and $v_j = x^{b_j}$?

$d = 7$: $x^0 \quad x^1 \quad x^3$ shows $Q(f) \geq 3$.

$x^0 \quad x^4 \quad x^3$

$c_i := -a_i - b_i \in \mathbb{Z}/2d\mathbb{Z} \rightsquigarrow$

$(a_i)_i, (b_j)_j, (c_k)_k$ satisfy $a_i + b_j + c_k = 0 \Leftrightarrow i = j = k$;

Aaronson-Green's *triangle removal lemma* $\Rightarrow r = o(d)$.

A case study: $K[x]/(x^d)$

Upper bound. $Q(f) \leq \lceil d/2 \rceil$: w.l.o.g. u_1 invertible $\rightsquigarrow$ $(u_i v_i)_{i>1}, (u_1 v_j)_{j\geq 1}$ linearly independent $\rightsquigarrow$ $2r - 1 \leq d$.

Proposition [Strassen, Chiu-D-Eggermont]
Actually, $Q(f) = \lceil d/2 \rceil =: e$, provided $|K| \geq e$.

Proof. $A := K[x]_{<e} + (x^d)$; $t_1, \ldots, t_e \in K$; $p := \prod_{i=1}^{e}(x - t_i)$
$\varphi : K[x]/(x^d) \to K[x]/(p), f + (x^d) \mapsto f + (p), \deg(f) < d$.

- φ is linear, $\varphi|_A$ is surjective, and
- $\varphi(ab) = \varphi(a)\varphi(b)$ for $a, b \in A$.

$\rightsquigarrow [\varphi, A, A]$ is a *partial epimorphism* to $K[x]/(p) \cong (K^e, I_e)$
$\rightsquigarrow Q(f) \geq e$. □

$Q(K[x_1, \ldots, x_n]/\mathfrak{m}^d)$: monomial? partial epimorphisms?

This talk

1. Generic subrank (after Derksen-Makam-Zuiddam)
2. Generic and maximal *border* subrank
3. *Symmetric* (border) subrank

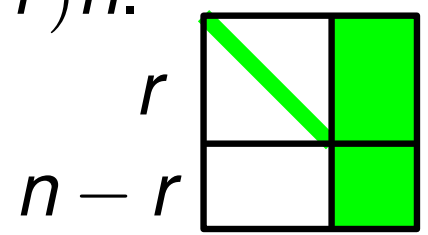
(4. *Real* versus *complex* subrank)

Theorem [Derksen-Makam-Zuiddam]
For $K = \overline{K}$ and $U = V = W = K^n$ and f sufficiently general, $Q(f) \approx \sqrt{3n}$.

Remark. For order-d tensors, $\approx Cn^{1/(d-1)}$. Exact values by [Pielasa-Šafránek-Shatsila].

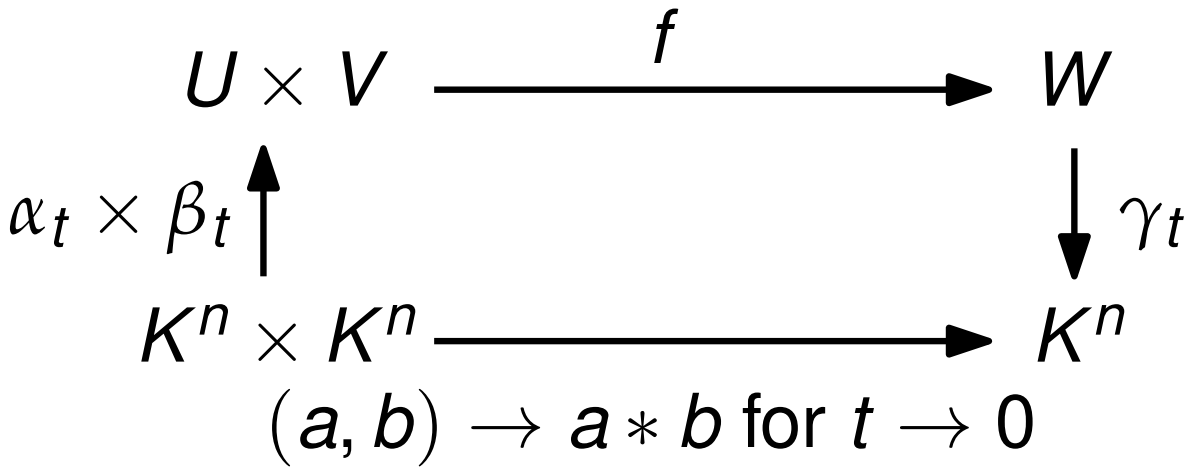
Proof sketch of $\leq$ • After acting with $\mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{GL}_n$, f with $Q(f) \geq r$ looks like $\widetilde{f}((a_1, a_2), (b_1, b_2)) = (\underbrace{a_1 * b_1}_{r}, \underbrace{\ldots}_{n-r})$.

• The affine space of such $\widetilde{f}$ has dim $n^3 - r^3$ and is stabilised by a group of dim $2r + 3(n-r)n$.



• So $\{f \mid Q(f) \geq r\}$ has dim
$\leq n^3 - r^3 + 3n^2 - 2r - 3(n-r)n$
$= n^3 - r(r^2 - 3n + 2) \rightsquigarrow$ density requires $r \leq \sqrt{3n-2}$. □

2. Border subrank



Definition. The *border subrank* of f is
$\underline{Q}(f) := \max\{r \mid \exists \alpha_t, \beta_t, \gamma_t : \lim_{t\to 0} \gamma_t \circ f \circ (\alpha_t \times \beta_t) = I_r\}$.

Theorem [Biaggi-Chang-D-Rupniewski]
For $K = \overline{K}$ and $U = V = W = K^n$ and f sufficiently general,
$\underline{Q}(f) \approx C\sqrt{n}$, and $C \geq 2 (> \sqrt{3})$.

Will discuss a related theorem on *maximal* border subrank.

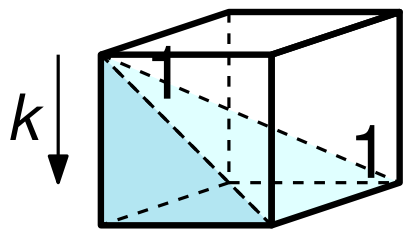
Maximal border subrank

$K = \overline{K}$ and $U = V = W = K^n \rightsquigarrow Q(f) \leq \underline{Q}(f) \leq n$.

$\{f \mid Q(f) = n\} = (\mathrm{GL}_n)^3 \cdot I_n$ has dimension $3n^2 - 2n$.

Proposition [Chang]

$\{f \mid \underline{Q}(f) = n\}$ has dimension $\geq \frac{2}{3}n^3 - O(n^2)$.



Proof

- Take $f := \overbrace{\sum_{i=1}^{n} e_i^{\otimes 3}}^{I_n} + \sum_{(i,j):i>k \text{ or } j>k} f_{ijk}\, e_i \otimes e_j \otimes e_k$.

- Define one-parameter subgroups $\lambda(t) = \mathrm{diag}(t^{2^1}, \ldots, t^{2^n})$ and $\mu(t) = \mathrm{diag}(t^{-2^2}, \ldots, t^{-2^{n+1}})$ in GL_n.

- Then $\lambda(t) \otimes \lambda(t) \otimes \mu(t) f \to I_n$ for $t \to 0$. □

Maximal border subrank

Theorem [Chang-D-Schnieper-Zuiddam]

$\{f \mid \underline{Q}(f) = n\}$ has dimension $= \frac{2}{3}n^3 - O(n^2)$.

Generalised Hilbert-Mumford Criterion [B-C-D-R]

G reductive $\curvearrowright X$ and $p, q \in X$ with $q \in \overline{Gp}$. Then $\exists \widetilde{q} \in Gq$ and $\lambda : K^* \to G : \lim_{t\to 0} \lambda(t)p = \lim_{t\to\infty} \lambda(t)\widetilde{q}$.

• Take $G = (\mathrm{GL}_n)^3$, $X = K^n \otimes K^n \otimes K^n$, $q = I_n$: if f has $\underline{Q}(f) = n$, then $\exists \lambda : K^* \to (\mathrm{GL}_n)^3$ such that $\lim_{t\to 0} \lambda(t)f = \lim_{t\to\infty} \lambda(t)\widetilde{q}$ for some $\widetilde{q} \in \mathrm{GL}_n^3 \cdot I_n$.

• Decompose $X = X_{<0} \oplus X_0 \oplus X_{>0}$ according to exponent of t with which $\lambda(t)$ acts: $f = f_0 + f_{>0} \in X_0 \oplus X_{>0}$ and $\widetilde{q} = \widetilde{q}_{<0} + \widetilde{q}_0 \in X_{<0} \oplus X_0$ and $\widetilde{q}_0 = f_0$.

A combinatorial lemma

• After a coordinate change,
$X_{\leq 0} = \langle e_i \otimes e_j \otimes e_k \mid a_i + b_j + c_k \leq 0 \rangle_K$ for $a, b, c \in \mathbb{Z}^n$.

• **Fact:** $\widetilde{q} \in X_{\leq 0} \cap (\mathrm{GL}_n)^3 I_n$ is *semi-stable* for $(\mathrm{SL}_n)^3$
$\rightsquigarrow D := \{(i,j,k) \mid a_i + b_j + c_k \leq 0\}$ supports a fractional perfect matching.

Lemma [Claude Sonnet 5, Bill Sands]
This implies that $|D| \geq \frac{1}{3} n^3$.

$\rightsquigarrow \leq \frac{2}{3} n^3$ free parameters in $f_{>0}$; $O(n^2)$ in $f_0 = \widetilde{q}_0$. □

Generalisation. Let $X_1, \ldots, X_d$ be random variables with $\mathrm{Prob}(\sum_i X_i \leq 0) = 1$; and $Y_1, \ldots, Y_d$ independent copies. Then $\mathrm{Prob}(\sum_i Y_i \leq 0) \geq \frac{1}{d}$.

3. Symmetric subrank

Definition
For $f \in K[x_1, \dots, x_n]_d$ we set $Q_s(f) :=$
$\max\{r \mid \exists \varphi \in \mathrm{End}(K^n) : \varphi f = x_1^d + \cdots + x_r^d\}$.

Theorem [Biaggi-D-de Nooij-van Santen]
For $K = \overline{K}$, sufficiently general f satisfies $Q_s(f) \approx (d!n)^{1/(d-1)}$.

Proof sketch for $\geq$: take $f = x_1^d + \cdots + x_r^d + \sum_{i=r+1}^n x_i f_i$ where $f_1, \dots, f_{n-r} \in K[x_1, \dots, x_r]_{d-1}$ sufficiently general.

$\dim(\langle x_i f_j \mid i, j = 1, \dots, r\rangle) = \min\{r(n-r), \dim K[x_1, \dots, x_r]_d\}$ by [Hochster-Laksov]. Hence if $r(n-r) >$binom$(r+d-1, d)$, then a map parameterising the $Q_s \geq r$ locus is dominant into $K[x_1, \dots, x_n]_d$. $\square$

Symmetric border subrank

Definition

$\underline{Q}_s(f) = \max\{r \mid \exists \varphi_t : \lim_{t\to 0} \varphi_t f = x_1^d + \cdots + x_r^d\}$

$Q_s \leftrightsquigarrow \underline{Q}_s$ are closer than $Q \leftrightsquigarrow \underline{Q}$:

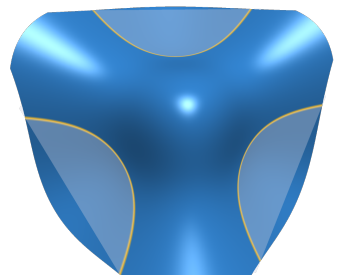
Observation

For $f \in K[x_1, \ldots, x_n]_d$ we have $Q_s(f) = n \Leftrightarrow \underline{Q}_s(f) = n$: if $I_n \in \overline{\mathrm{GL}_n f}$, then $\dim \mathrm{GL}_n I_n \leq \dim \mathrm{GL}_n f$, but the left-hand side is already n^2.

Theorem [Biaggi-D-de Nooij-van Santen]

If $d = 3$ and $\underline{Q}_s(f) \leq 3$, then $\underline{Q}_s(f) = Q_s(f)$.

(Similar result for $\underline{Q}_s(f) \leq 2$ and $d = 4$, but false for $d = 5$.)



4. Real versus complex subrank

$K = \mathbb{R}$. Then $f : U \times V \to W \rightsquigarrow f_{\mathbb{C}} : U_{\mathbb{C}} \times V_{\mathbb{C}} \to W_{\mathbb{C}}$, and $Q(f) \leq Q(f_{\mathbb{C}})$.

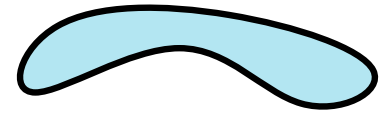
Example

$U = V = W = \mathbb{C}$ as $\mathbb{R}$-spaces

$f(a, b) = ab \rightsquigarrow Q(f) = 1$ but $Q(f_{\mathbb{C}}) = 2$.

$\Delta > 0, Q = R = 2$

$\Delta = 0$



$\Delta < 0, Q = 1, R = 3$

Theorem [Biaggi-D-Eggleston]

For arbitrary $f : U \times V \to W$, $Q(f) \geq \sqrt{Q_{\mathbb{C}}(f)}$.

Zuiddam with ChatGPT: this is optimal up to a constant!

Theorem [B-D-E]

The Hadamard product $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, seen as a $2n$-dimensional $\mathbb{R}$-algebra, has $Q(f) = n$ while $Q_{\mathbb{C}}(f) = 2n$.

Conclusion

Well understood:

- Generic (border) subrank and its symmetric analogues.
- The discrepancy between subrank over $\mathbb{R}$ and over $\mathbb{C}$.
- Special cases ($\mathbb{C}^n$ as $\mathbb{R}$-algebra, cubics with low $\underline{Q}_s$, ...)

Techniques:

- Partial epimorphisms.
- Geometric invariant theory.
- Combinatorics (triangles, fractional matchings).
- Commutative algebra (syzygies).

Open

- Value of $Q(f)$ for algebras such as $K[x_1, \ldots, x_n]/\mathfrak{m}^d$?
- How to compute $Q(f)$ for arbitrary f?

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