Catalan-many morphisms to trees, part I

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(University of Bern, TU Eindhoven)
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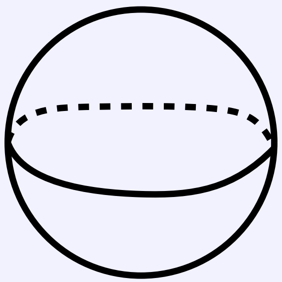
Frankfurt, TGiZ, April 24, 2020

Definition

Riemann surface: a compact complex manifold X of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} =$ 

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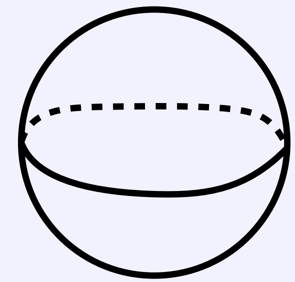

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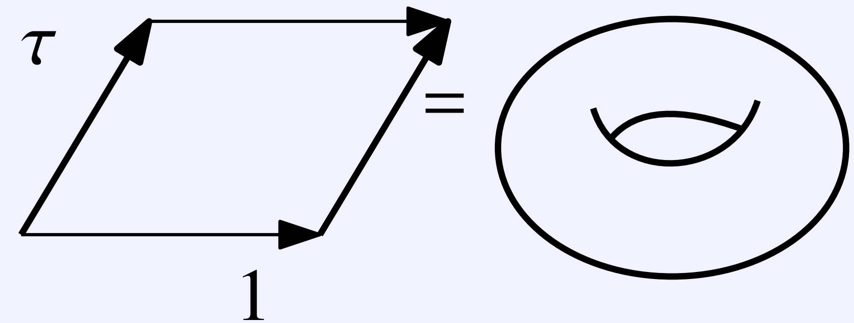

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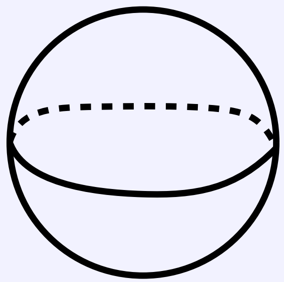

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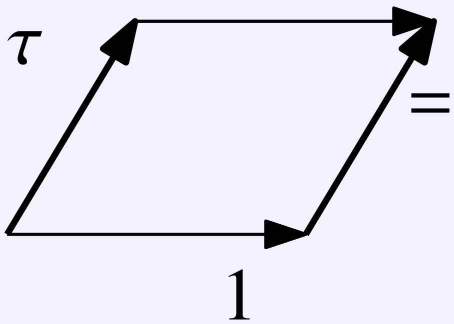

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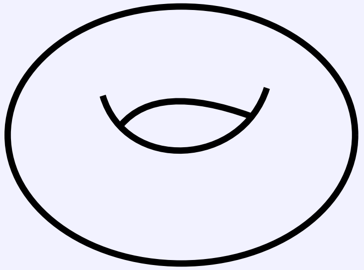

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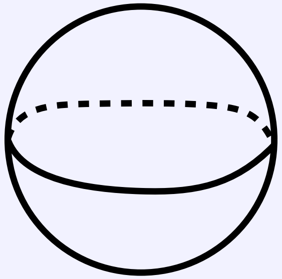

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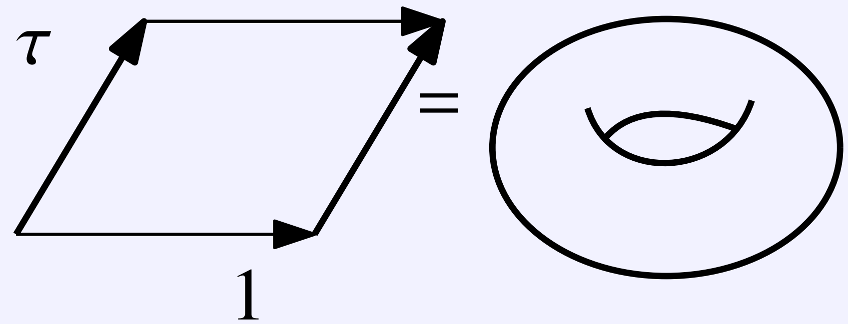

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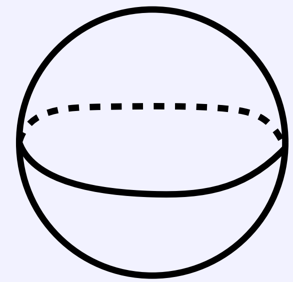

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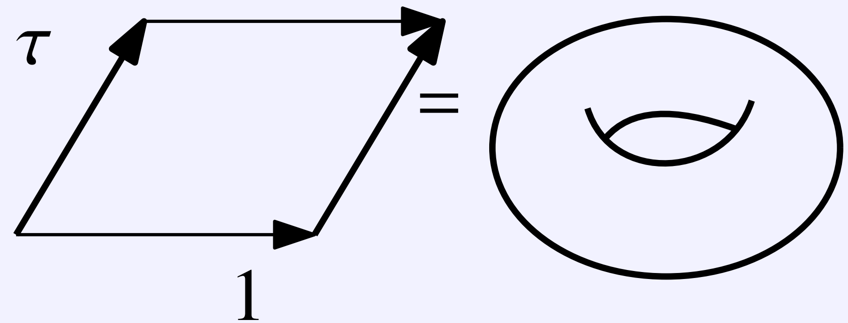

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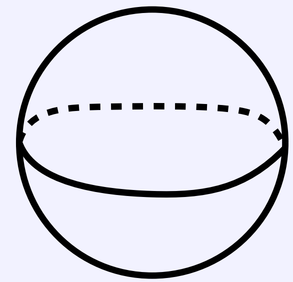

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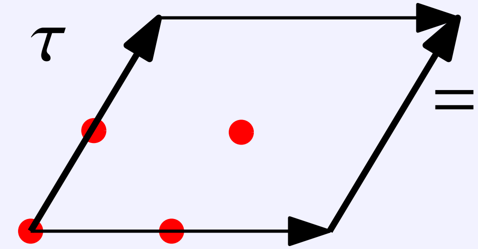

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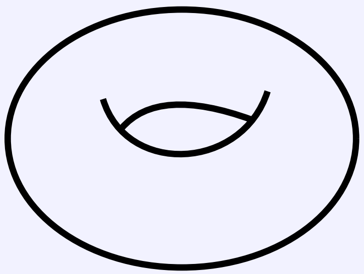

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gonality 2:

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m+n\tau)^2} + \frac{1}{(m+n\tau)^2} \right)$$

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The gonality theorem for Riemann surfaces

Theorem [Riemann,Kempf,Kleiman-Laksov,Griffiths-Harris,. . .]

• Any X of genus g has gonality at most $1 + \lceil g/2 \rceil$.

• Equality for X sufficiently general.

• For g even, a sufficiently general X has precisely $C_{g/2}$ such holomorphic maps (up to PGL_2).

$C_{g/2} = 1, 2, 5, \ldots$ for $g = 2, 4, 6, \ldots$ is the $g/2$-th *Catalan number*.

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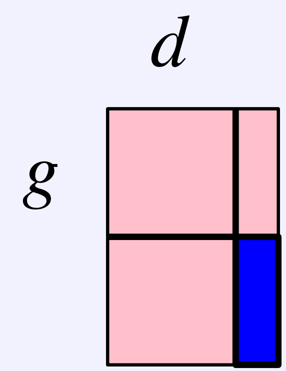

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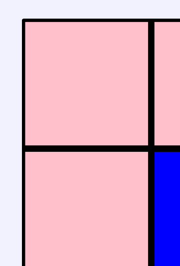
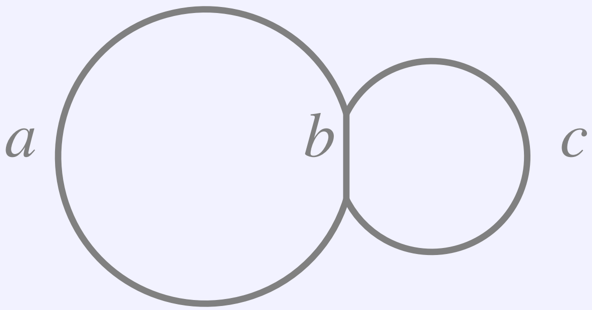


Expected dimension of W^1_d: $d - (g - (d - 1)) - 1$; want ≥ 0.

Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex Γ with a suitable metric:



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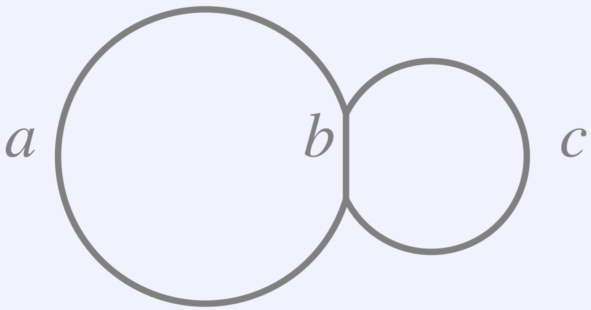

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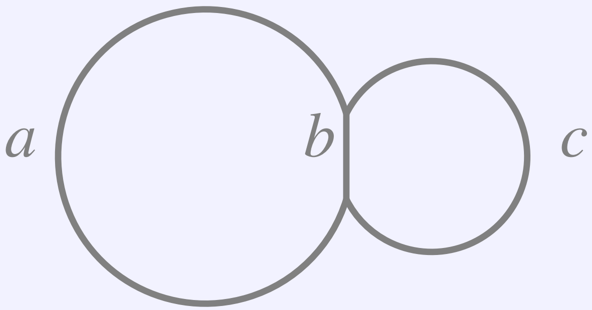

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A continuous $\varphi : \Gamma \to \Sigma$ is *harmonic* if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and e, f emanating from $\varphi(v)$ we have

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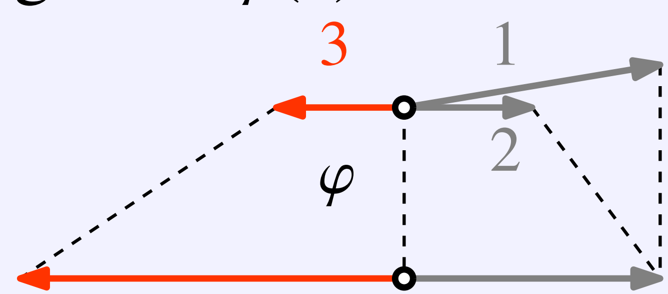

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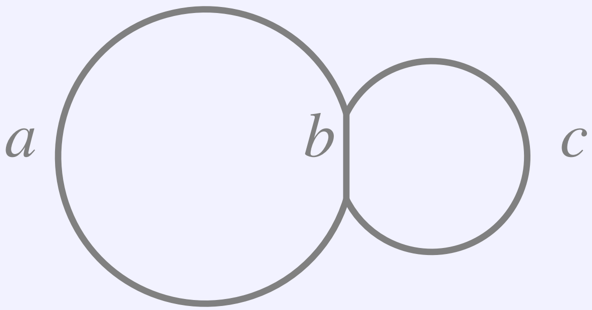

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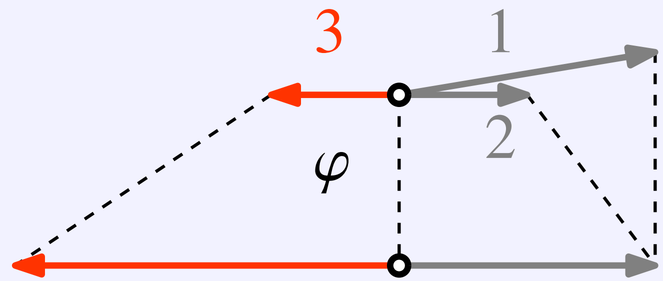

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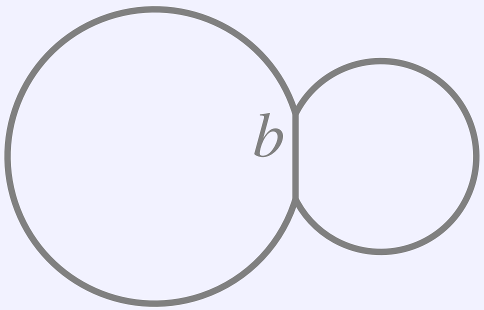

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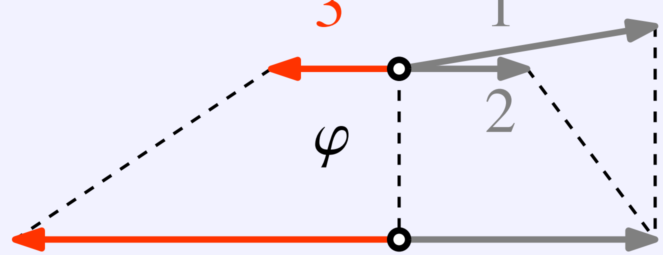

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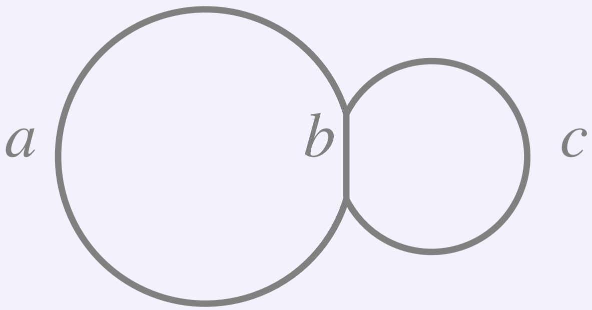

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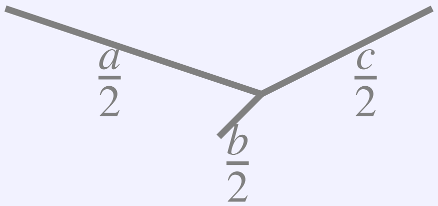

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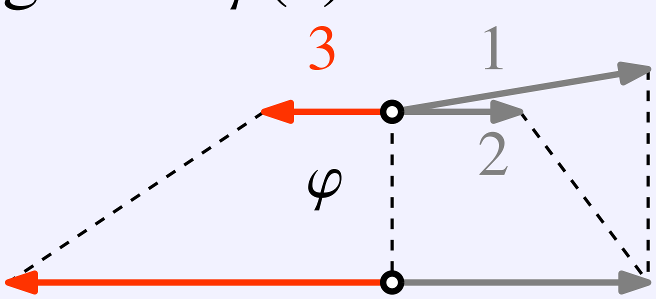

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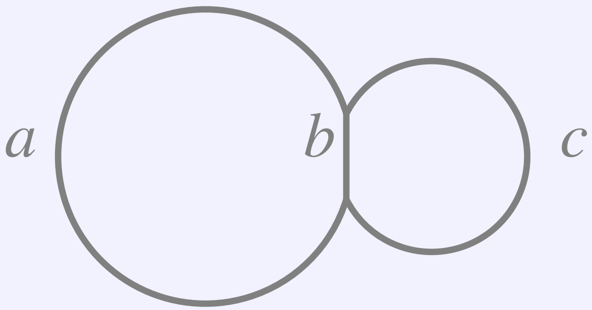

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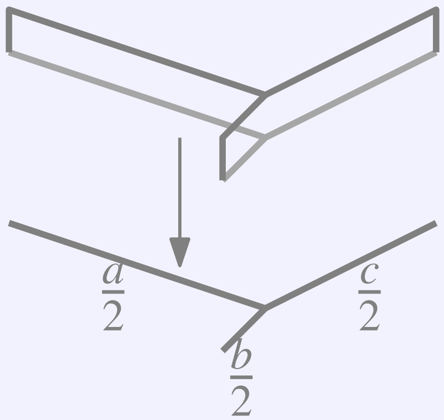

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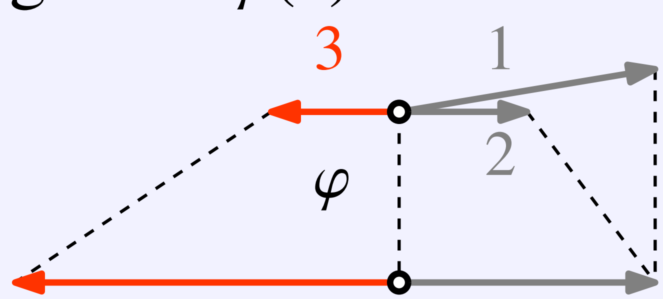

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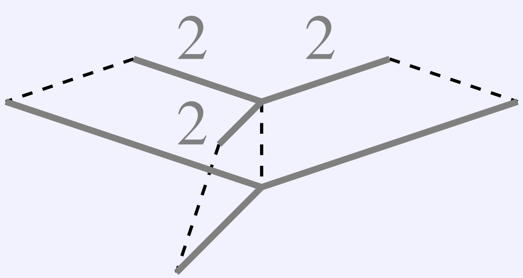

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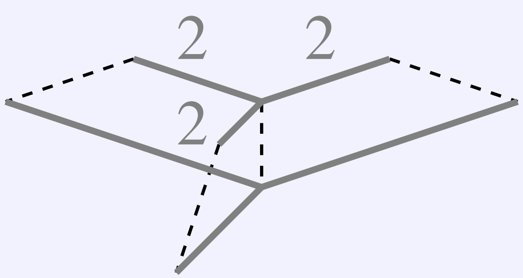

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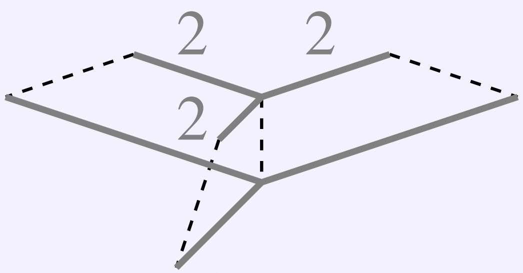

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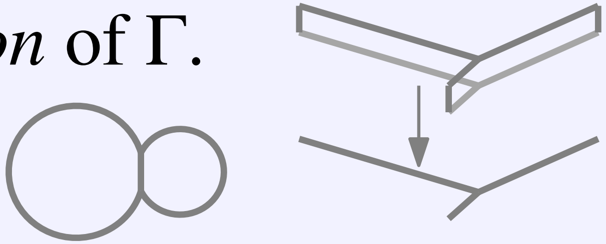
In the example, the gonality is 2*:*



The gonality theorem for metric graphs

Theorem [Baker,Caporaso,. . . ,D-Vargas]

• Any Γ of genus g has gonality at most $1 + \lceil g/2 \rceil$.

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- D-Vargas is independent of these, and completely combinatorial (but $\sim$ 80 pages).

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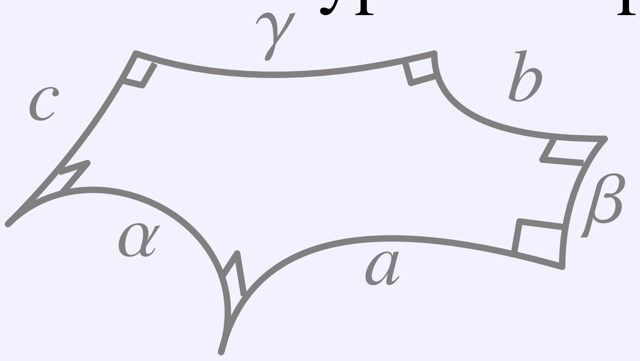
Plan for part I: Discuss the relation between two theorems, and part of Cools-D. Part II (Alejandro): more combinatorics.

Approximating metric graphs

(We follow recent work by Lionel Lang, which extends older work by Mikhalkin.)

Lemma from hyperbolic geometry: given $\alpha, \beta, \gamma > 0$ there exists a unique right-angled hexagon in the hyperbolic plane with side lengths $\alpha, a, \beta, b, \gamma, c$.



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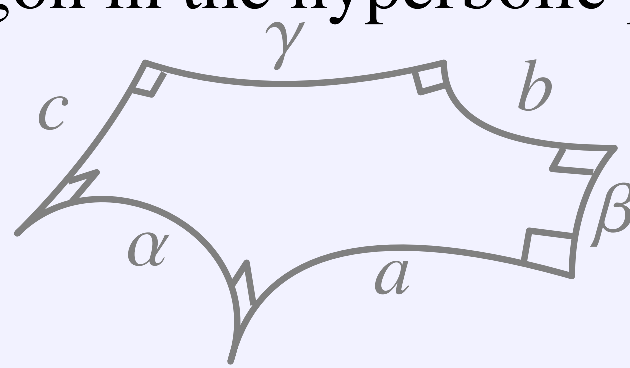
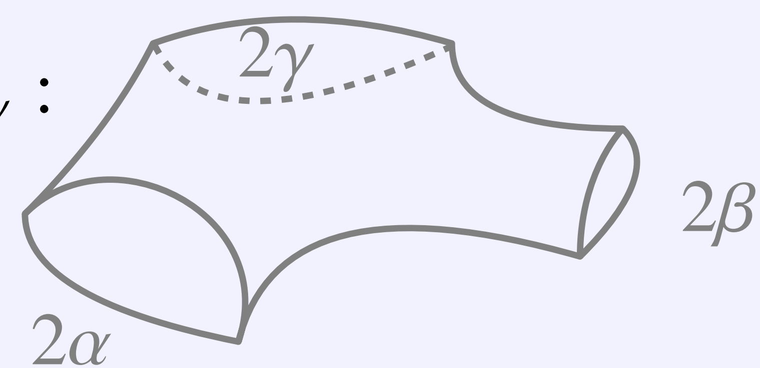


Glue two copies to a *pair of pants* $P_{2\alpha,2\beta,2\gamma}$:



Approximating metric graphs, continued

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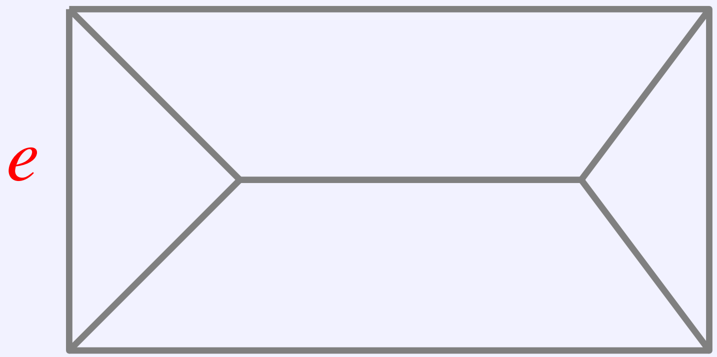

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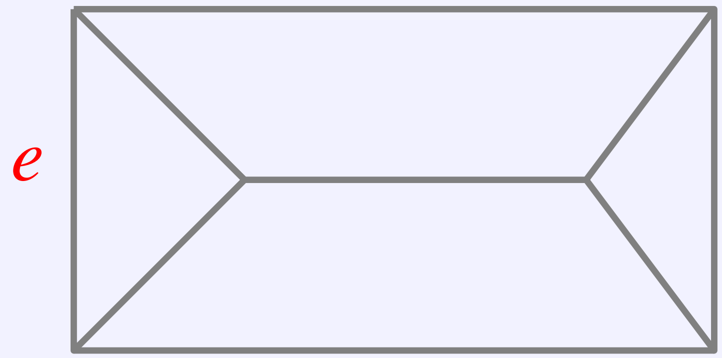
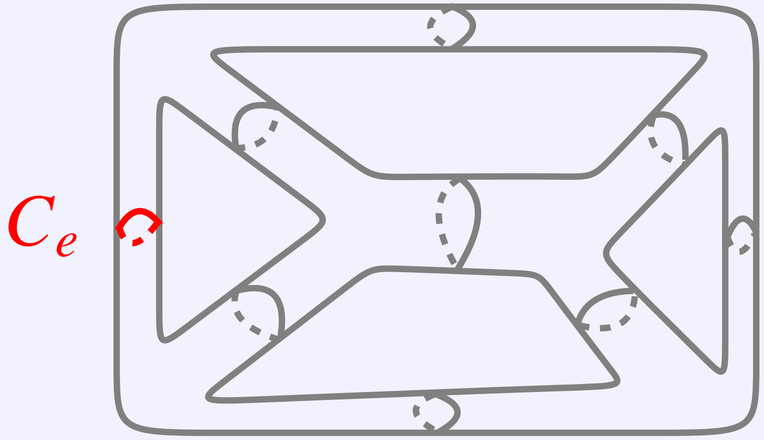


For each $v \in V$ incident to e_1, e_2, e_3, take a copy P_v of $P_{c(e_1),c(e_2),c(e_3)}$, and glue these to a Riemann surface X_c of genus g:



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Let $\psi_t : X_t \to \mathbb{P}^1$; can be chosen to depend continuously on t.

Theorem [Mikhalkin,...,Lang]

The ψ_t converge in a well-defined sense to a tropical morphism from a modification of Γ to a tree.

Why tree and modification?

For $t \gg 0$, the images $\psi_t(C_e) =: \tilde{C}_e$ in $\mathbb{P}^1$ are disjoint.

Simplifying assumption: they are topological circles in $\mathbb{P}^1$.

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Create graph T with $V(T)$ =connected components of $\mathbb{P}^1 \setminus \bigcup_{e \in E(G)} \tilde{C}_e$; an edge if they have a common $\tilde{C}_e$ in their boundary. Since $\mathbb{P}^1$ is simply connected, T is a tree.

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In the limit, we find a tropical morphism from a modification Γ' of Γ with combinatorial type G' to a metric tree with combinatorial type T.

Why balancing and Riemann-Hurwitz?

Pick a vertex $v \in V(G')$; this corresponds to a connected component U of $X_t \setminus \bigcup_{e \in E(G')} C_e$.

Assume no loops at v. Then $\overline{U}$ is $\mathbb{P}^1$ minus k discs corresponding to the edges incident to v; Euler characteristic: $2 - k$.

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For each $\tilde{C}_e, e \in E(G)$ in the boundary of $\overline{W}$ and $C_{e'}, e' \in E(G')$ in the boundary of $\overline{U}$ mapping onto $\tilde{C}_e$, let $m_{e'}$ be the degree of that map $C_{e'} \to \tilde{C}_e$. These are the slopes in our tropical morphism; $m_\varphi(v)$ is the degree of the branched cover $\psi_t|_{\overline{U}} : \overline{U} \to \overline{W}$.

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R-H formula: $2 - k = m_\varphi(v)(2 - l) - \sum_{p \in U}(e_p - 1)$; so $(k - 2) \geq m_\varphi(v) \cdot (l - 2)$

Moduli space of genus-g metric graphs

- Let $g \geq 2$.
- For each ordinary genus-g graph $G = (V, E)$ set $C_G := (\mathbb{R}_{>0})^E$.
- For any isomorphism $G \to H$ glue C_G to C_H.
- If contracting e in G yields a genus-g graph H, glue C_H to C_G as the boundary with e-th coordinate 0.
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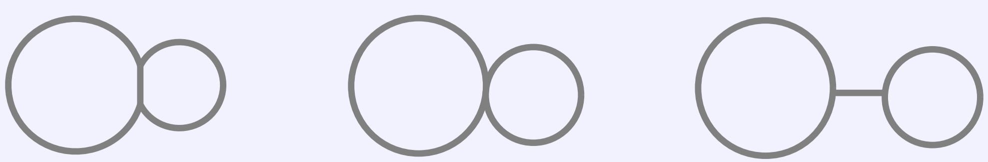
If G trivalent, then $\dim C_G = |E| = 3g - 3 \rightsquigarrow \dim M_g = 3g - 3$.



Caporaso: M_g connected in codimension 1.

Older combinatorial results by Cools-D

Theorem

For $d, g \geq 2$ the gonality-d locus in M_g is locally closed of dim $\min\{3g-3, 2g+2d-5\}$ (*perhaps not pure-dim*). In particular, the locus where the gonality is $\geq 1 + \lceil g/2 \rceil$ is dense and open.

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Remarks • Dimension matches the classical count for curves.
• Via approximation, the first Theorem implies that a general genus-g Riemann surface has gonality (at least) $1 + \lceil g/2 \rceil$ — no need for a *specific* graph to prove this. (*Observed by Mikhalkin in 2011.*)

Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).
• Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
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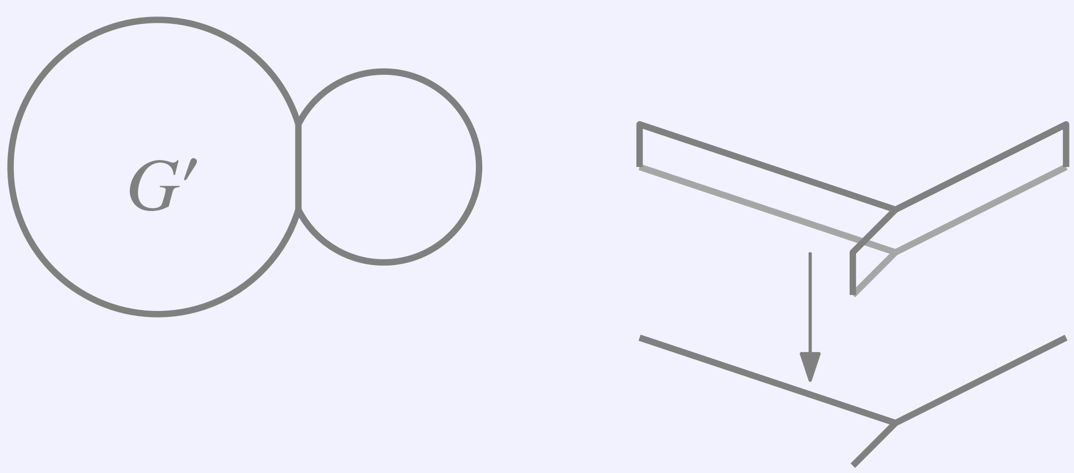

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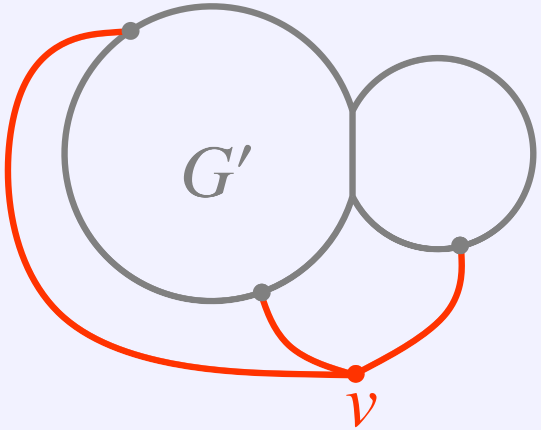

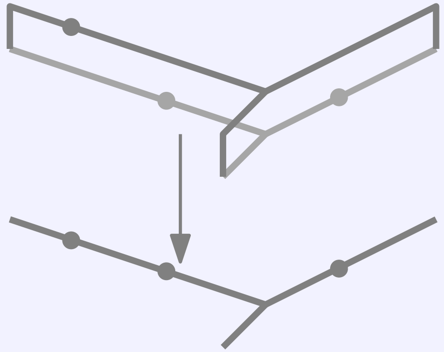

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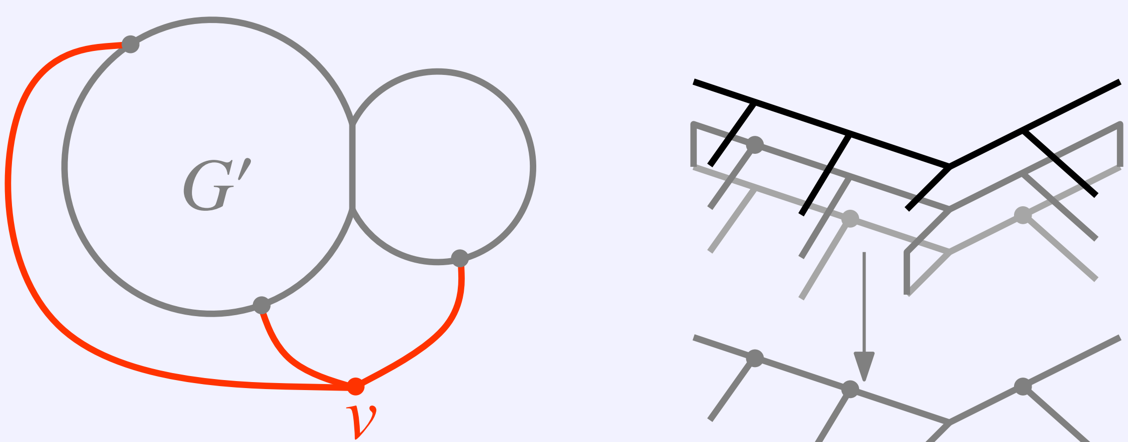

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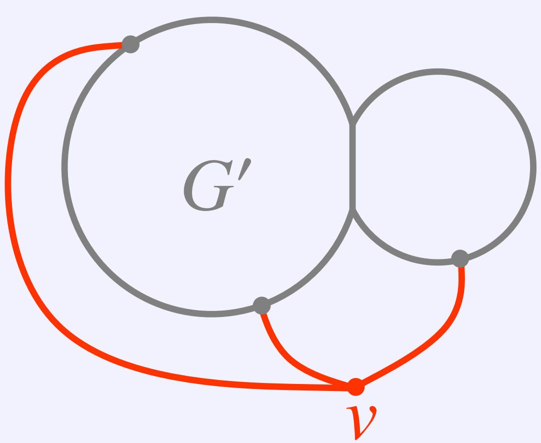

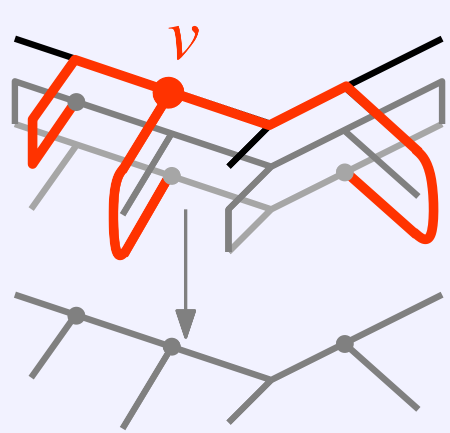

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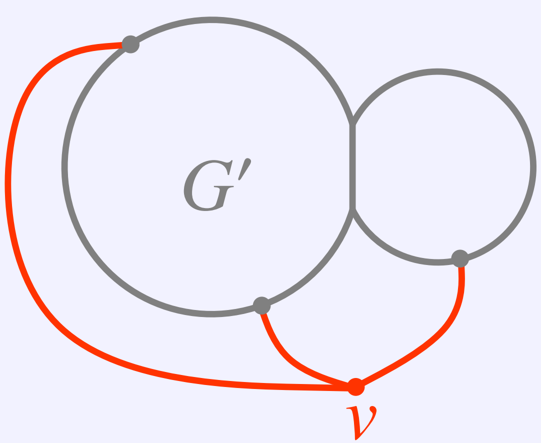
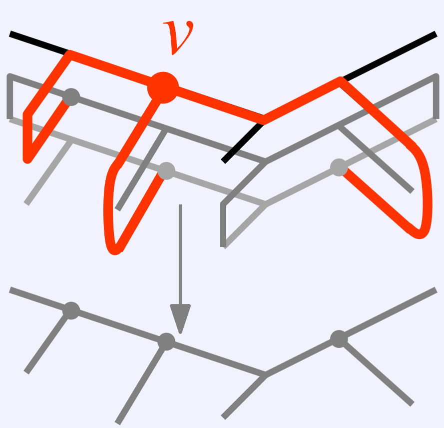
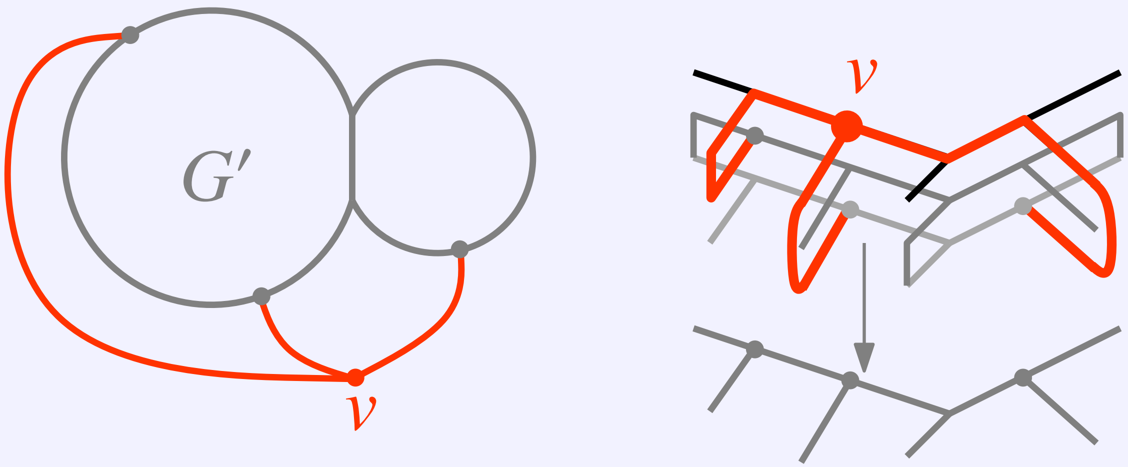




Parameter count:
3 for the gray dots
3 for the orange edges
$3g - 9 + 3 + 3 = 3g - 3$ $\square$

The Cools-Draisma method: remarks and limitations



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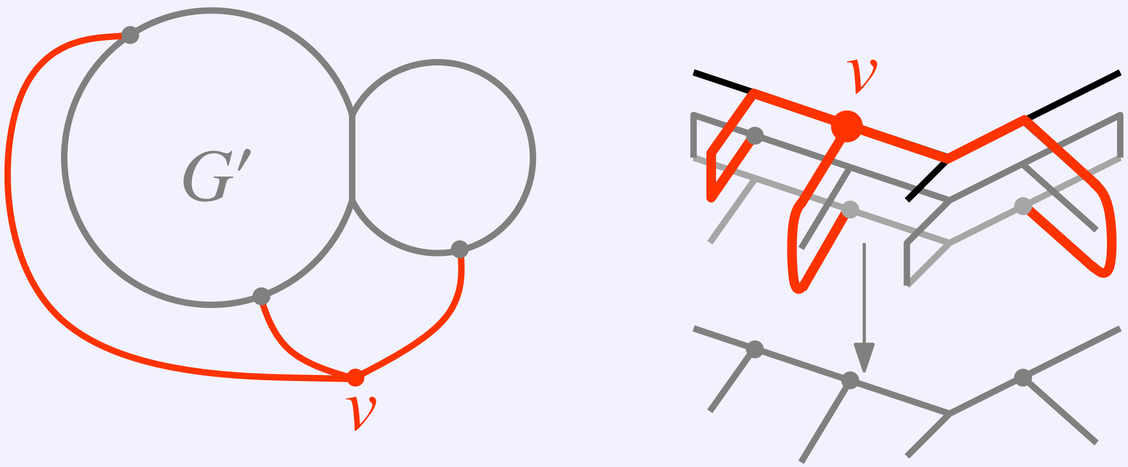

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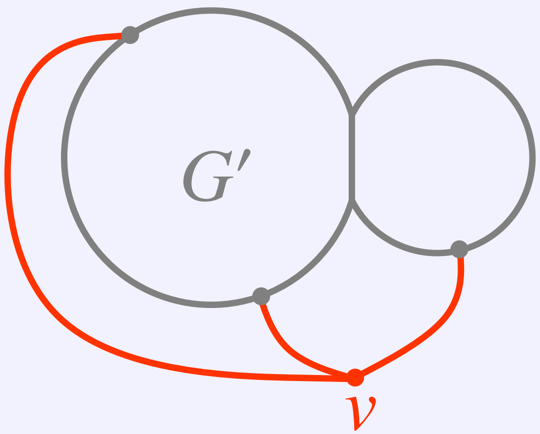

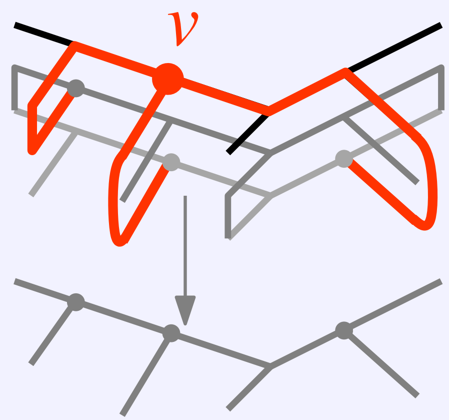

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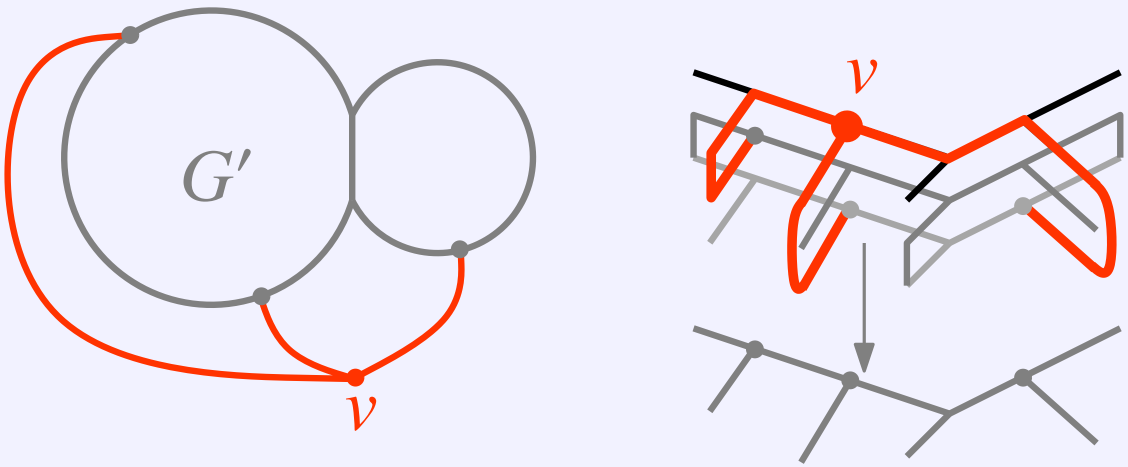

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Answer: YES, see Alejandro's talk next!