Finite up to symmetry

Jan Draisma
TU Eindhoven

(with Rob Eggermont, Jochen Kuttler, . . .)

Mathematisches Kolloquium, Frankfurt, June 2012

Hilbert's Basis Theorem

David Hilbert (1862-1943)

$f_1(x_1, \ldots, x_n) = 0$
$f_2(x_1, \ldots, x_n) = 0$
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reduces to a *finite* system.

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Bruno Buchberger (1942–)

Gröbner bases, algorithmic methods.

(Non-)Noetherian rings

Example

$x_1 = 0, x_2 = 0, x_3 = 0, \ldots$

does not reduce.

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$\Rightarrow \exists j$ with $f_j \in R f_1 + R f_2 + \ldots + R f_{j-1}$.

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$K[x_1, x_2, \ldots, x_n]$ Noetherian,

$K[x_1, x_2, \ldots]$ not Noetherian,

. . . but it *is up to symmetry!*

Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_1, \alpha_2, \ldots \in \mathbb{Z}^n_{\geq 0}$
$\Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{Z}^n_{\geq 0}$.

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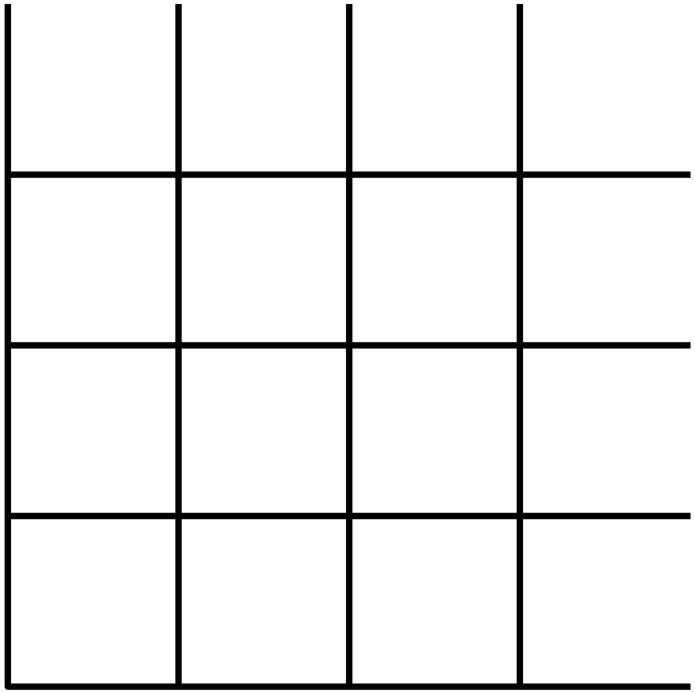

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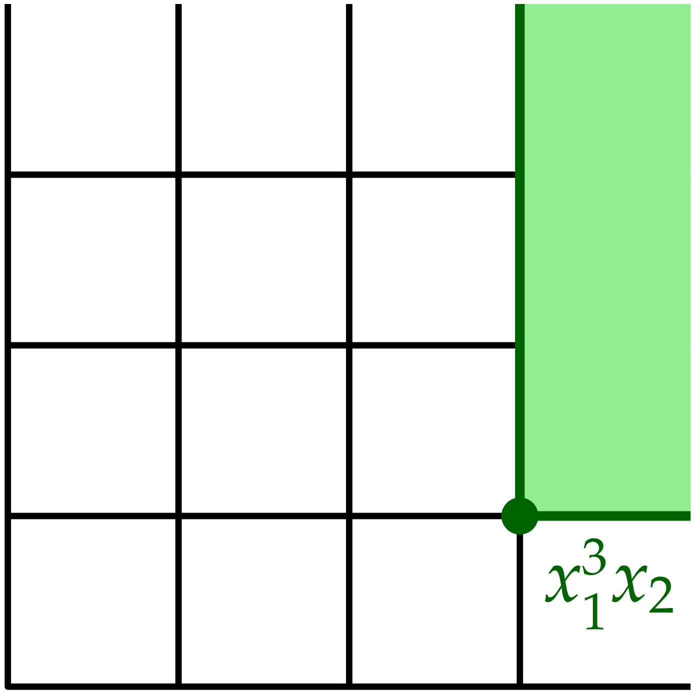

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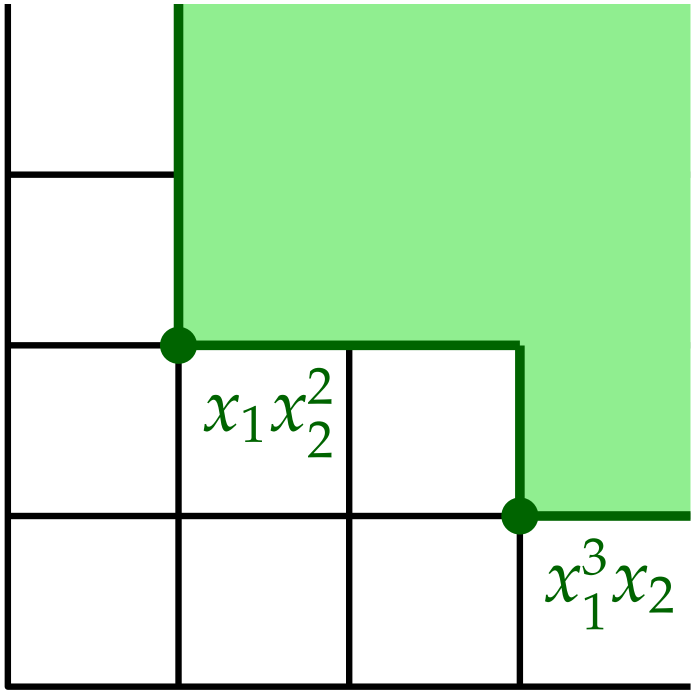

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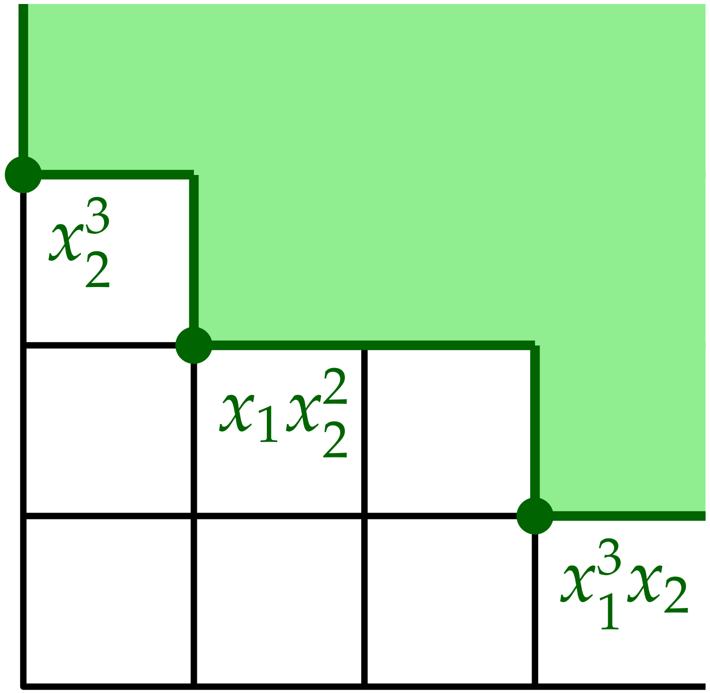

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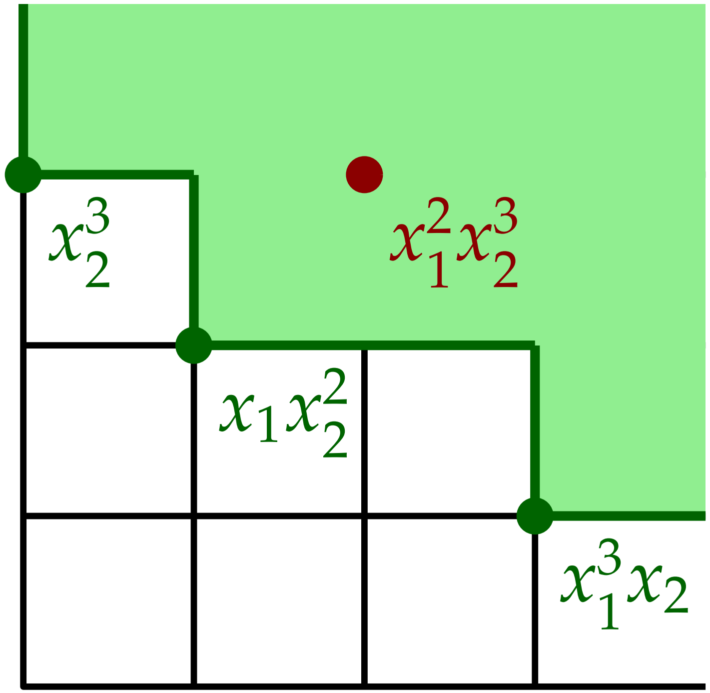

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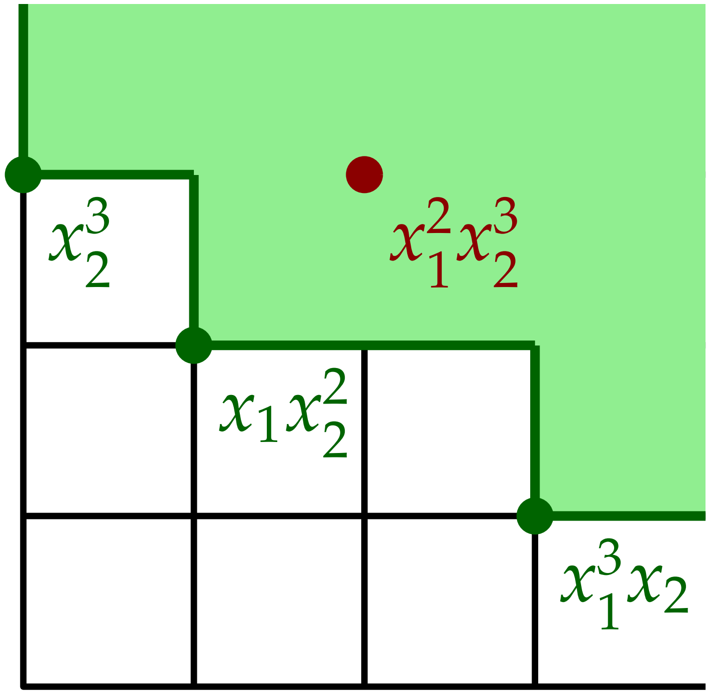

Dixon's Lemma $\Rightarrow$ Hilbert's Basis Theorem

Well-partial-orders

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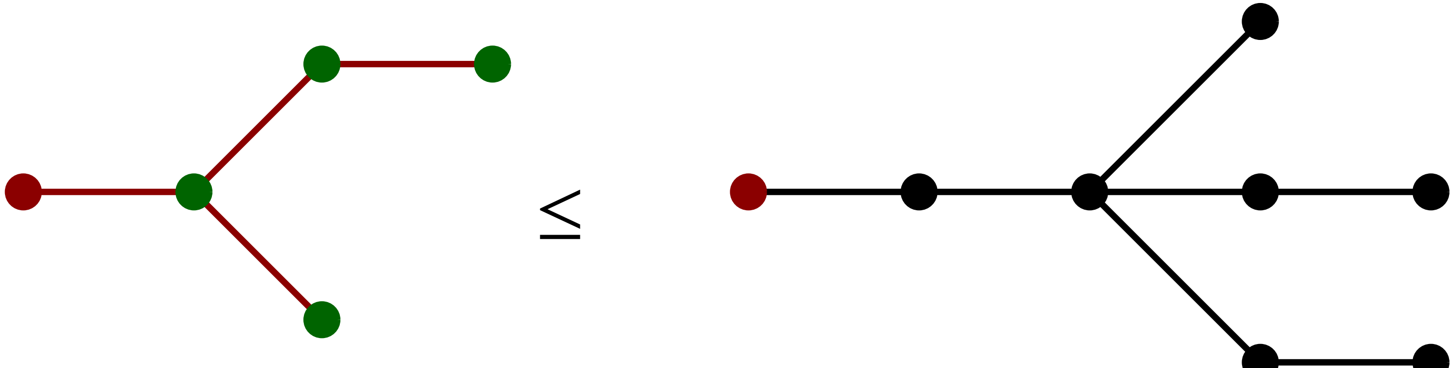
Examples

$\mathbb{N}$, $\mathbb{N}^n$ (Dixon's Lemma)

Rooted trees (Kruskal's Tree Theorem)



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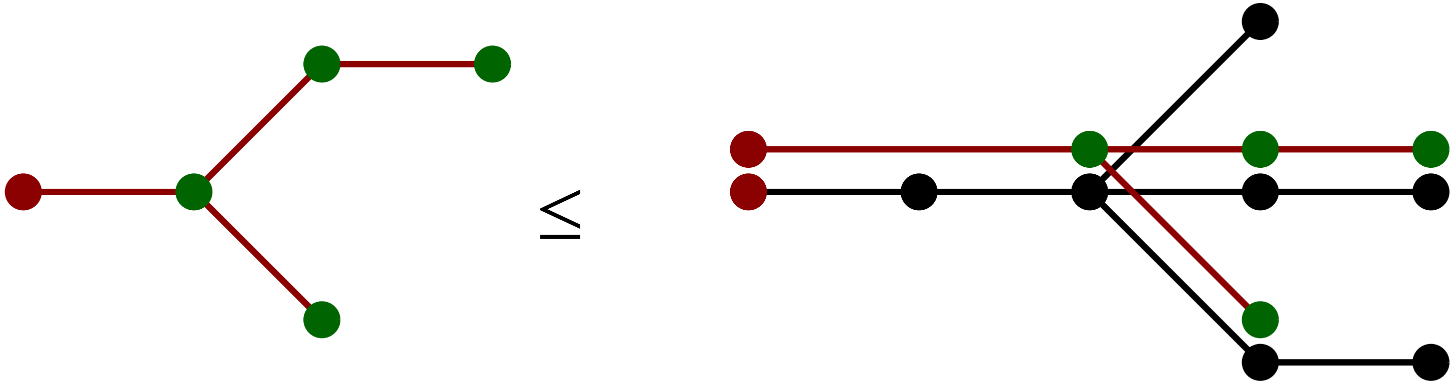

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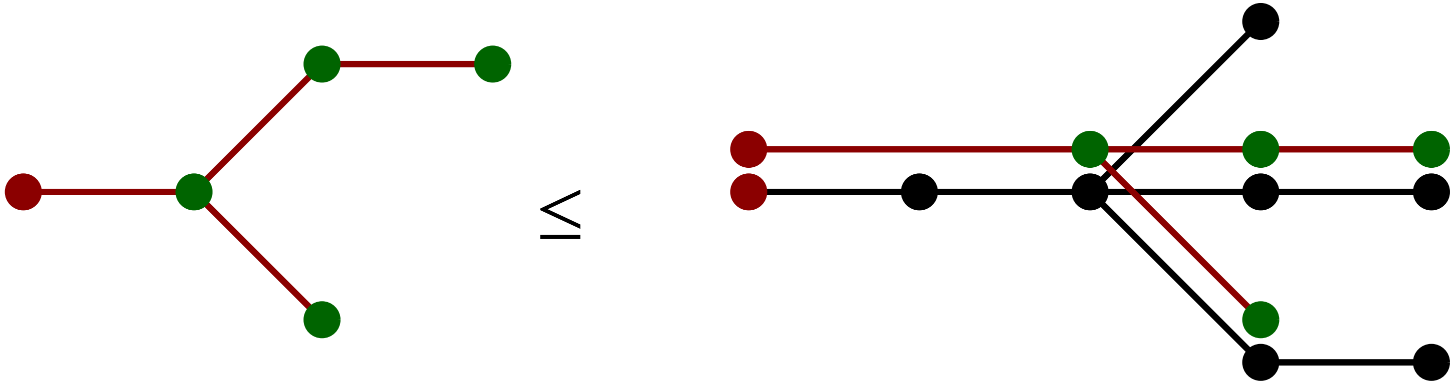

Graphs with the minor order!

(Robertson-Seymour)

Higman's Lemma

Graham Higman (1917-2008)
$(S, \leq)$ well-partial-order
then so is $(S^*, \leq)$ with
$(s_1, \ldots, s_k) \leq (t_1, \ldots, t_l) :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$.

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$S = [n]$ finite, $\leq$ trivial
$\rightsquigarrow w_1, w_2, \ldots \in S^*$
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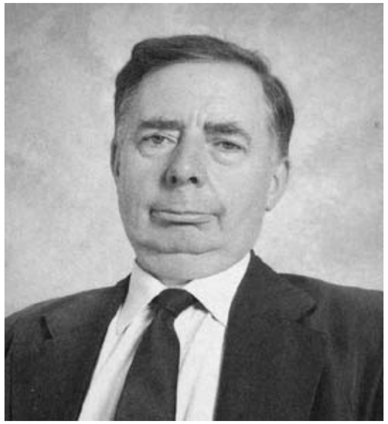
Crispin Nash-Williams (1932–2001)
Short proof.



Back to algebra

Crucial example

$S = \mathbb{N}^n$, $\leq$ as in Dixon's lemma

$S^* \to \{\text{monomials in } x_{ij},\ i = 1, \ldots, n,\ j \in \mathbb{N}\}$

$$x_{11} \cdot x_{12}x_{22}^2 \cdot x_{23} \cdot x_{14} =: u$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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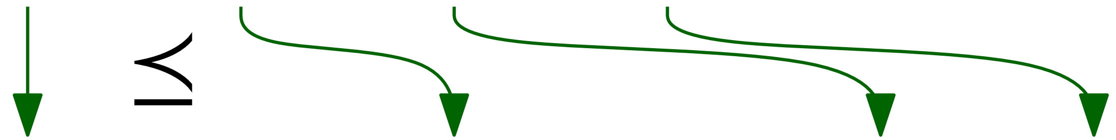

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$$\pi = \begin{array}{c} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 5 \\ 4 \mapsto 6 \\ \cdots \end{array}$$

$\pi \in \mathrm{Inc}(\mathbb{N})$

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$\pi \in \mathrm{Inc}(\mathbb{N})$

$\mathrm{Inc}(\mathbb{N})$ acts on monomials:

$\pi x_{ij} := x_{i\pi(j)} \rightsquigarrow \pi u$ divides v.

Noetherian up to symmetry

Higman's Lemma $\rightsquigarrow$
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$R := K[x_{ij}]$ with $\mathrm{Inc}(\mathbb{N})$-action

Theorem (Cohen 87, Hillar-Sullivant 09)
$f_1, f_2, \ldots \in R \Rightarrow \exists p$ with
$f_p \in R \cdot \mathrm{Inc}(\mathbb{N}) f_1 + \ldots + R \cdot \mathrm{Inc}(\mathbb{N}) f_{p-1}$

Corollary
R is *Noetherian up to* $\mathrm{Sym}(\mathbb{N})$.

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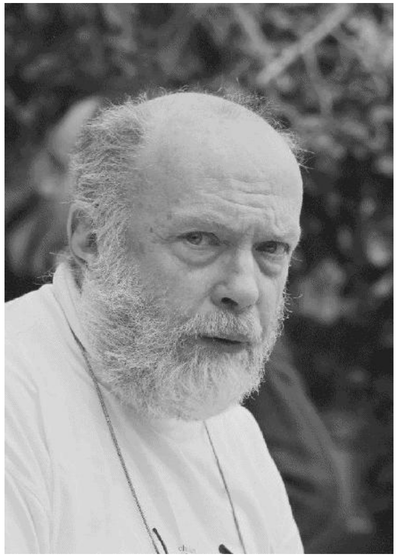
Conclusion
$K^{[n]\times\mathbb{N}}$ is Noetherian up to $\mathrm{Sym}(\mathbb{N})$.

I: Vandermonde relations

Andreas Dress (1938-)

$y_1, y_2, \ldots$ variables

$z_I := \prod_{i,j \in I, i<j}(y_i - y_j)$ for $|I| = n$, fixed



Relations among the z_I finite up to $\mathrm{Sym}(\mathbb{N})$*?*

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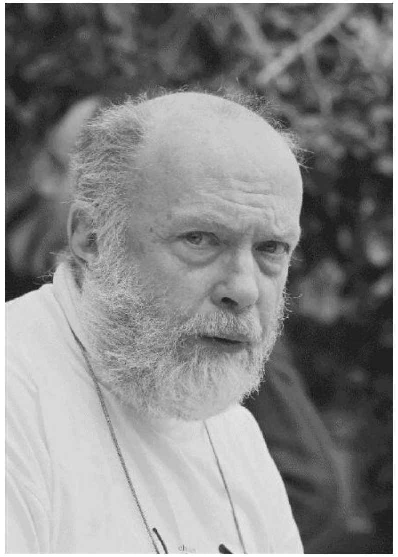

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Theorem (JD)

Yes (in characteristic zero).

$$z_I = \det \begin{bmatrix} 1 & \cdots & 1 \\ y_{i_1} & \cdots & y_{i_n} \\ \vdots & & \vdots \\ y_{i_1}^{n-1} & \cdots & y_{i_n}^{n-1} \end{bmatrix}$$ satisfy *Plücker relations*.

mod these $\rightsquigarrow$

invariant ring $K[x_{ij}, i \in [n], j \in \mathbb{N}]^{\mathrm{SL}_n}$; use Reynolds.

II: Independent Set Theorem

Row and column sums

$A, B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with $a_{i+} = b_{i+}$ and $a_{+j} = b_{+j}$

$\Rightarrow \exists A = A_0, A_1, \ldots, A_k = B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with

$$A_l - A_{l-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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No three-way interaction

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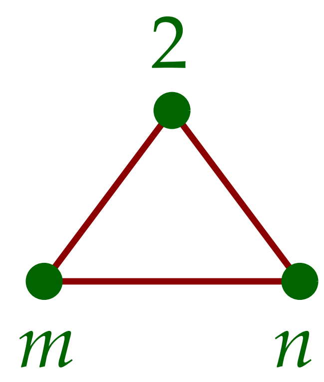
$\rightsquigarrow$ moves *grow* with as $m, n \to \infty$.



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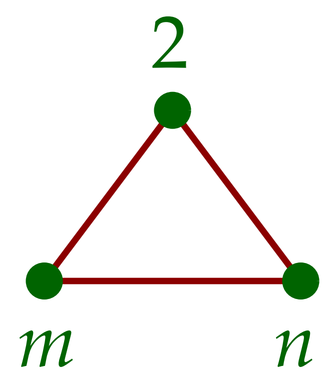

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Chris Hillar, Seth Sullivant

Sufficient condition for stabilisation.

III: Bounded-rank tensors

Definition
Rank of $\omega \in V_1 \otimes \cdots \otimes V_p$ is minimal k in $\omega = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{ip}$.

Border rank is minimal k such that $\omega \in \overline{\{\text{rank} \leq k \text{ tensors}\}}$.

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Flattening and contracting:
$v_1 \otimes \cdots \otimes v_p \mapsto (v_1 \otimes \cdots \otimes v_q) \otimes (v_{q+1} \otimes \cdots \otimes v_p)$
$v_1 \otimes \cdots \otimes v_p \mapsto x(v_p) \cdot v_1 \otimes \cdots \otimes v_{p-1}, \quad x \in V^*.$

III: Proof sketch

$X_p \subseteq V^{\otimes p}$ border rank $\leq k$
$Y_p \subseteq V^{\otimes p}$ all flattenings have rank $\leq k$

$x_0 \in V^*$

$$V^{\otimes 0} \xleftarrow{\langle \cdot , x_0 \rangle} V^{\otimes 1} \longleftarrow V^{\otimes 2} \longleftarrow \cdots \longleftarrow V^{\otimes \infty}$$

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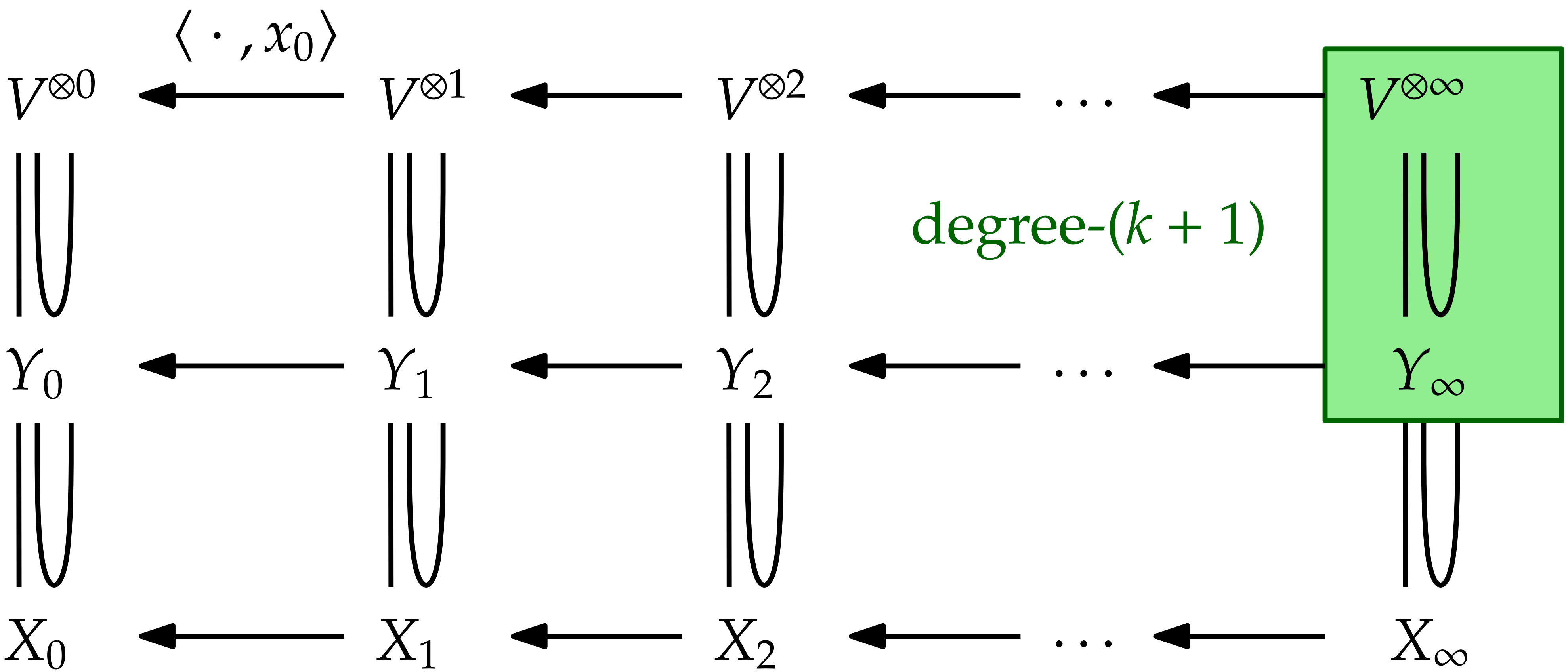

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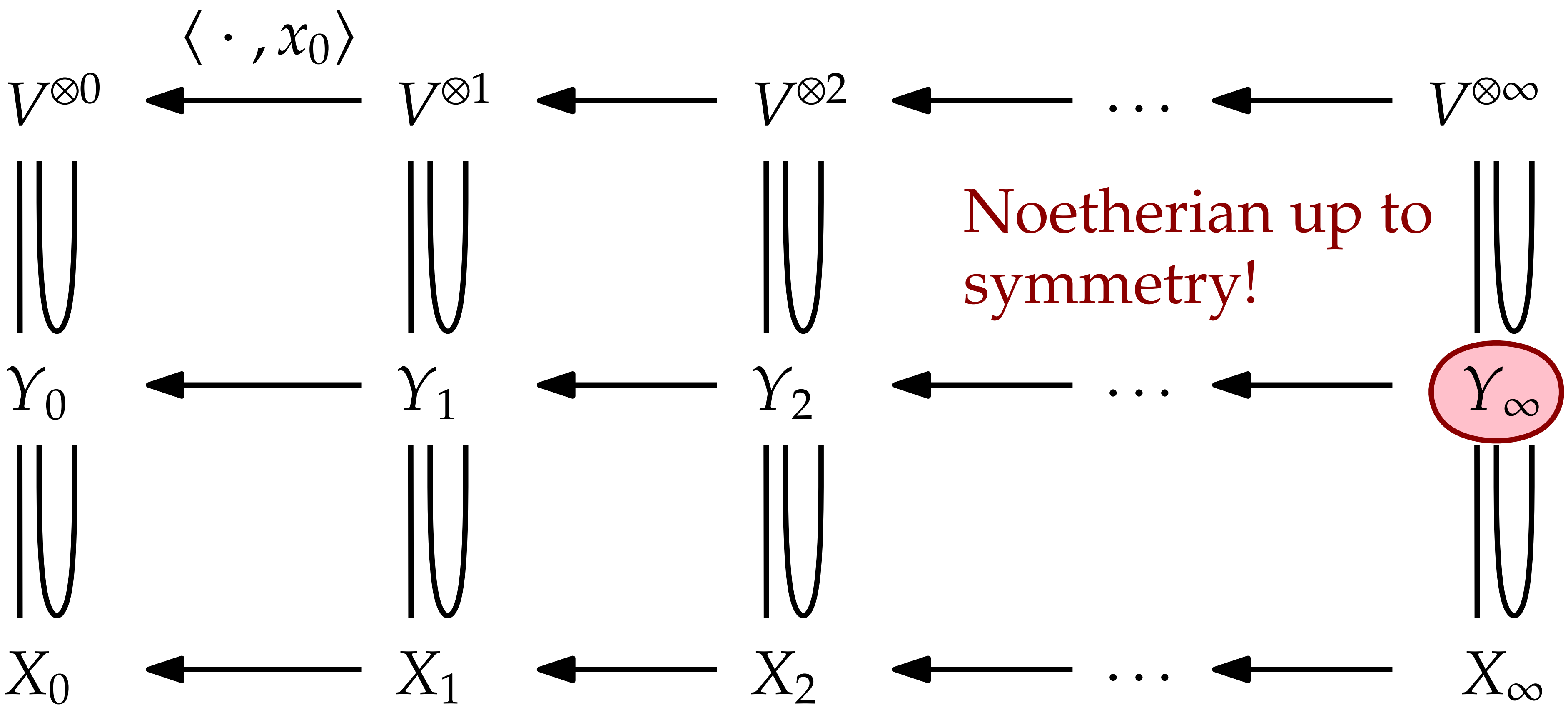

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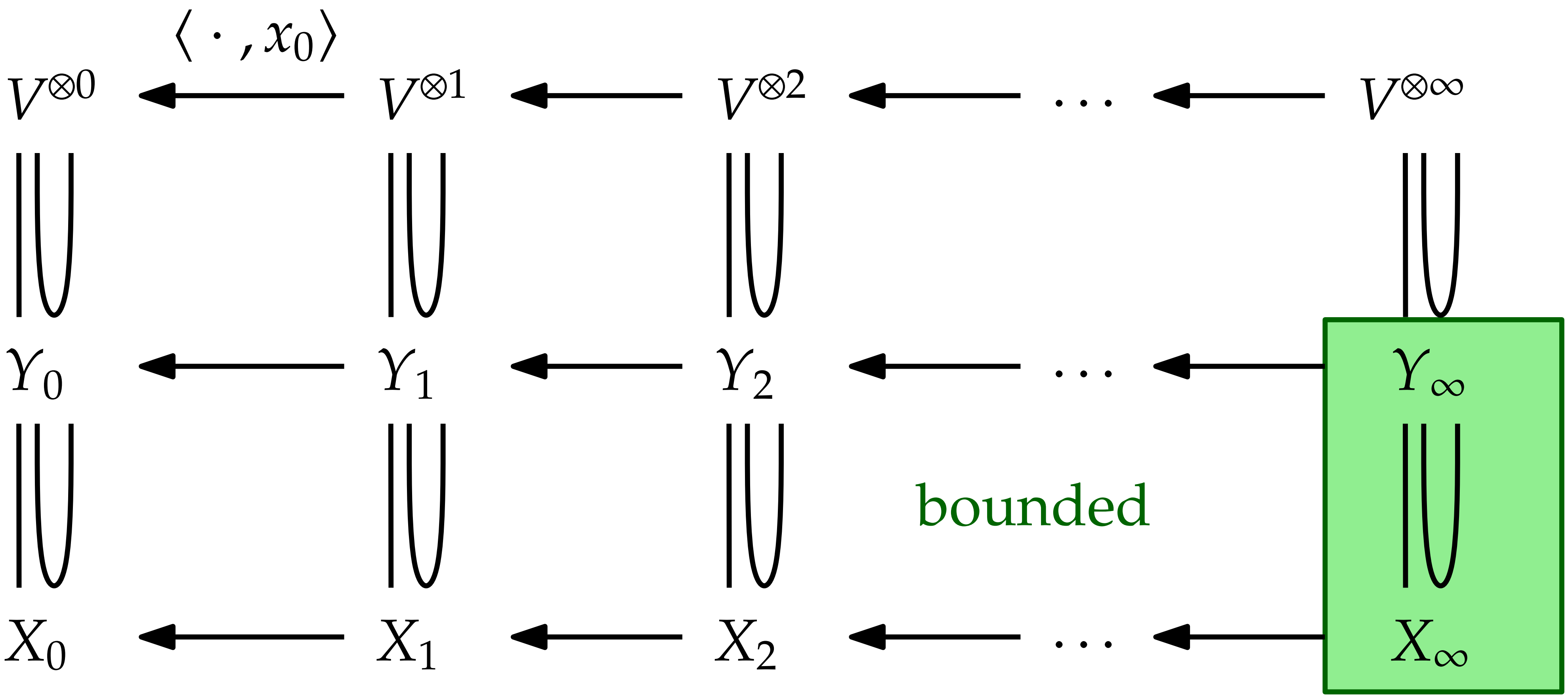

Further topics

Phylogenetic tree models
defined in bounded degree, independent of the tree (Rob Eggermont-JD).

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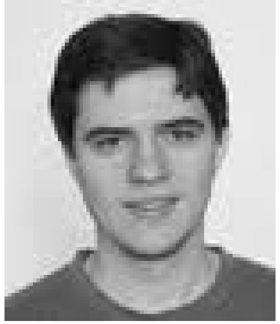
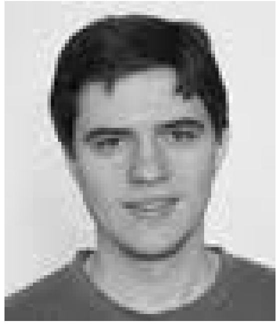
Syzygies of Segre (Andrew Snowden)
For each k, the Segre Embedding
$\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_p \to \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$
has finitely many types of syzygies.



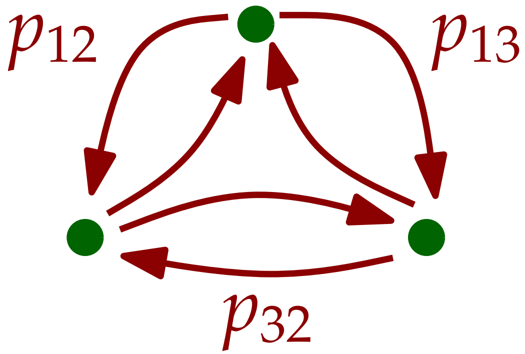
Further topics

Phylogenetic tree models
defined in bounded degree, independent of the tree (Rob Eggermont-JD).

Syzygies of Segre (Andrew Snowden)
For each k, the Segre Embedding
$\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_p \to \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$
has finitely many types of syzygies.



Markov chains (Ruriko Yoshida et al)



$\text{Prob}(1232) = p_{12}p_{23}p_{32}$

Relations among path probabilities stabilise as $T \to \infty$?

Further topics?

Skew-symmetric tensors
Inverse Grassmannian
Infinite wedge
Matroid minor conjecture

Symmetric tensors
Inverse Veronese
Infinite symmetric power

Computational issues

Any other?

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Any other?

Many infinite-dimensional varieties can be described with finitely many equations up to symmetry.