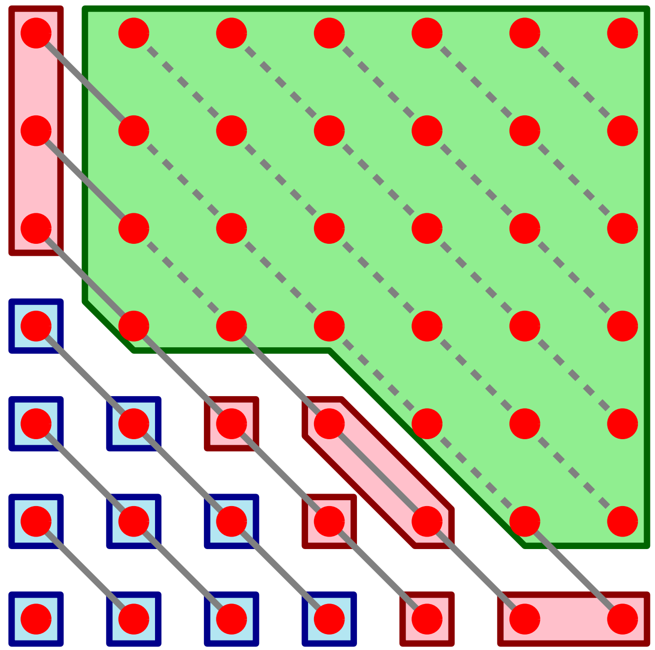
# The space of monomial preorders



George Metcalfe

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Simon Santschi

FoCM
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Jan Draisma
University of Bern

# Monoid preorders

$\Pi$ a commutative monoid, e.g. $\Pi = \{x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\} =:$ Mon$_n$

**Definition**

A *preorder* $\leq$ on $\Pi$:

- $(u \leq v$ and $v \leq w) \Rightarrow u \leq w$
- $u \leq u$
- $u \leq v$ or $v \leq u$
- $u \leq v \Rightarrow uw \leq vw$

we do *not* require:

- $(u \leq v$ and $v \leq u) \Rightarrow u = v$
- $1 \leq u$

**Example**

$(\Pi, \cdot) = (\mathbb{R}^k, +)$ with $\leq_{\text{lex}}$

**Theorem** [Robbiano, 1985]

Any preorder $\leq$ on $(\mathbb{Z}^n, +)$ is the pull-back of $\leq_{\text{lex}}$ under some homomorphism $\rho : \mathbb{Z}^n \to \mathbb{R}^k$, $k \leq n$.

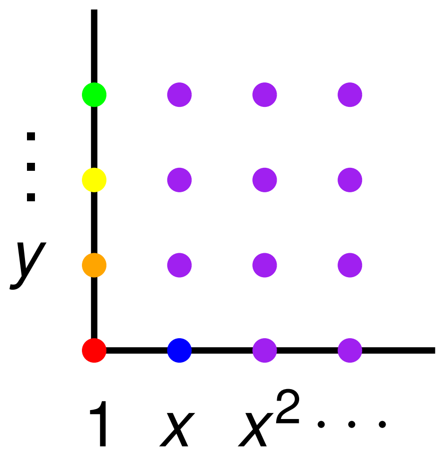
# Preorders on monomials



On $\mathrm{Mon}_n$ Robbiano's theorem fails:

**Example**

$a \leq b$ if $a$'s color is at least as high in the rainbow as $b$'s; e.g. $y \leq x$ and $xy \leq x^2$




But if $\leq = \rho^*(\leq_{\text{lex}})$, then
$xy \geq x^2 \Rightarrow \rho(x) + \rho(y) \geq \rho(x) + \rho(x)$
$\Rightarrow y \geq x$, a contradiction.

**Question**

Characterisation of preorders on $\mathrm{Mon}_n$ *à la* Robbiano?

# The space of preorders



**Notation**

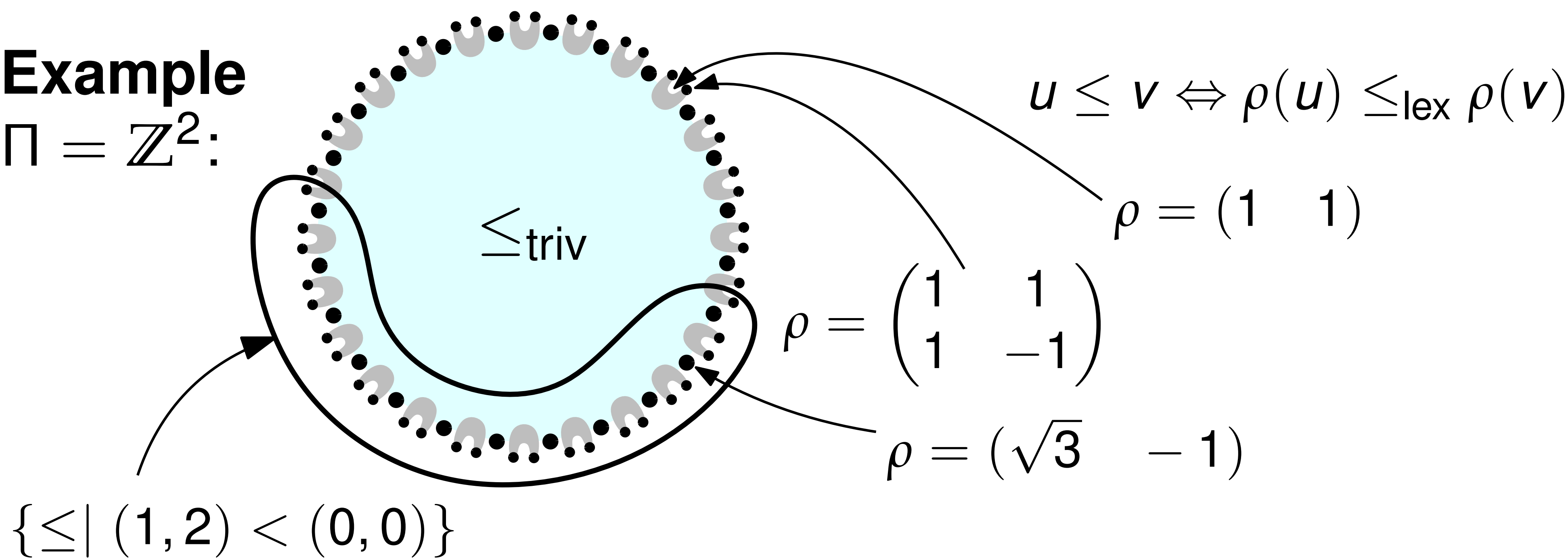
- $u \approx v$ means ($u \leq v$ and $v \leq u$)
- $u < v$ means ($u \leq v$ and not $v \leq u$)

**Definition**

$\mathcal{P}(\Pi) = \{$all preorders on $\Pi\}$; $\{\leq \mid u < v\}$ *closed* for all $u, v$

**Example**

$\Pi = \mathbb{Z}^2$:



# Spectrality



**Theorem A**

• $\mathcal{P}(\Pi)$ is an irreducible spectral topological space, and if $\Pi$ is f.g., then every point is open in its closure.

• For all $n \geq 1$, $\mathcal{P}(\mathbb{Z}^n)$ has Krull dimension $n$ but $\mathcal{P}(\text{Mon}_n) \cong \mathcal{P}(\mathbb{Z}^n_{\geq 0})$ has infinite Krull dimension.

**Example**

$\mathcal{P}(\mathbb{Z}_{\geq 0})$ :

$0 > 1 \approx 2 \approx 3 \approx \cdots$

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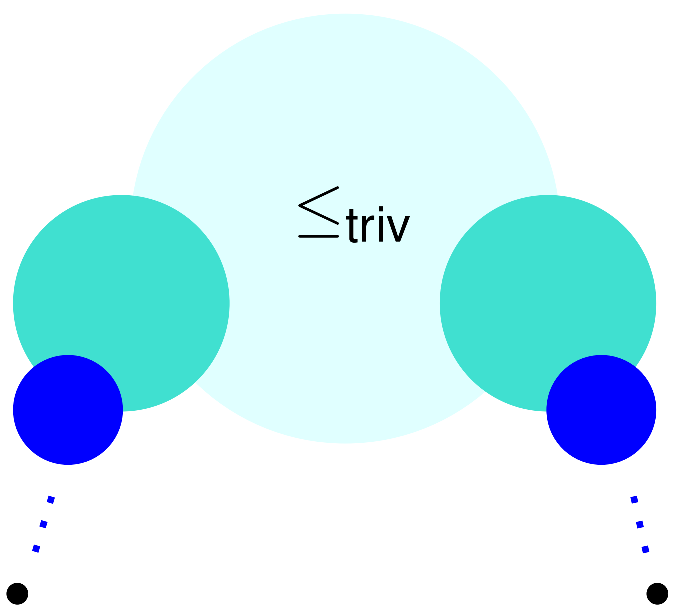


$0 < 1 \approx 2 \approx 3 \approx \cdots$

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$0 < 1 < 2 < 3 < \cdots$

**Definition**
An *admissible set* in $\mathbb{R}^N$ is $S \setminus H$ with $S$ semi-algebraic and $H$ a countable union of hyperplanes, both defined over $\mathbb{Q}$.

**Theorem B**
If $\Pi$ is finitely generated (a quotient of some $\mathrm{Mon}_n$), then $\mathcal{P}(\Pi)$ admits a countable cover by admissible sets.

**Theorem C**
The universal theory of ordered commutative monoids is decidable.

**Open problem**
A Chevalley theorem: given $\rho : \Pi_1 \to \Pi_2$ and a constructible set $C \subseteq \mathcal{P}(\Pi_2)$, is $\rho^*(C) \subseteq \mathcal{P}(\Pi_1)$ constructible?

# (Necessary) generalisation



**Definition**

- $\Pi$-*set* $S$: a set with an action $\Pi \times S \to S, (u, s) \mapsto us$
- preorders on $S$: $s \leq t \Rightarrow us \leq ut$ (no preorder on $\Pi$)

*Theorems A–C hold for* $\mathcal{P}_\Pi(S) = \{$*preorders on* $S\}$*, for any f.g.* $\Pi$ *acting with finitely many orbits on* $S$*.*

**Note:** $\Pi = \mathrm{Mon}_n$, $S = \mathrm{Mon}_n \times \{1, \dots, m\} \rightsquigarrow \mathcal{P}(S) = \{$monomial preorders for the free module $K[x_1, \dots, x_n]^m\}$

**Rest of this talk:**

1. $\Pi = \mathbb{Z}^n$: “Robbiano’s Theorem” for $\mathbb{Z}^n$-sets
2. the case $S = \Pi = \mathrm{Mon}_n$

# 1. Generalising Robbiano's theorem to $\Pi$-sets

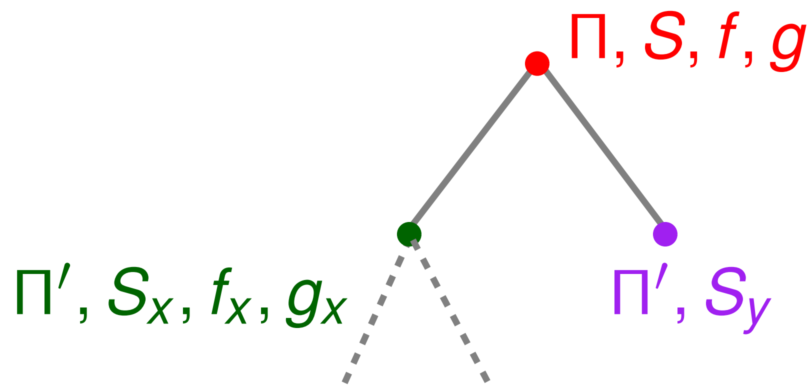


$\Pi = \mathbb{Z}^n$ acting on $S$ with finitely many orbits

**Proposition**
$\leq$ on $S$ nontrivial $\rightsquigarrow \exists f : \Pi \to \mathbb{R}$ homo, $g : S \to \mathbb{R}$ nonconstant s.t. $g(us) = f(u) + g(s)$ and $s \leq t \Rightarrow g(s) \leq g(t)$.

**Recurse:** $\Pi' := \ker(f)$, $X \subseteq \operatorname{im}(g)$ coset reps of $\operatorname{im}(f)$, $S_x := g^{-1}(x) \rightsquigarrow \leq$ is determined by $f$, $g$, and $\leq |_{S_x}$ on the $\Pi'$-sets $S_x$ for $x \in X$.

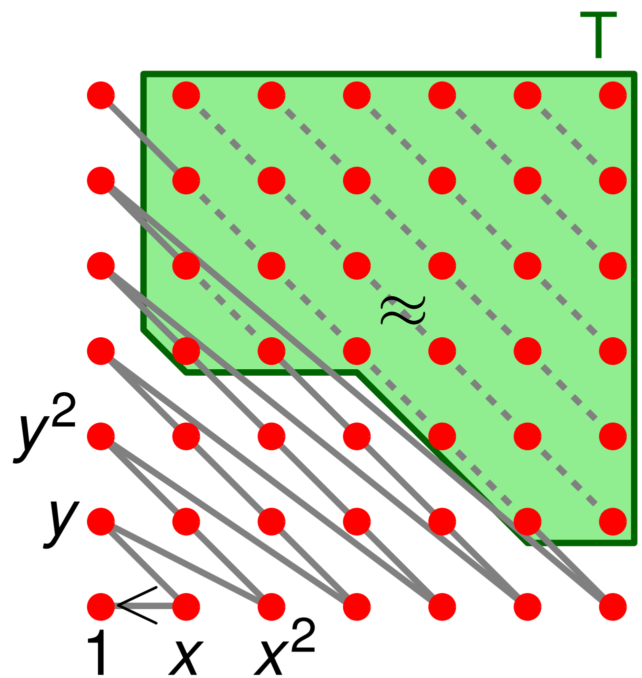
$\rightsquigarrow$ Get a *preotree* for $\leq$ with *numerical data* $(f_x, g_x)_x$ in some admissible space.




**Theorem** [DMS, Rust-Reid 97, ...]**:** Every preorder on $S$ comes from a preotree with numerical data, and vice versa.

$\leq$ a preorder on $S = \Pi = \mathrm{Mon}_n$
$\rightsquigarrow$ *coarsening* $\leq'$ defined by $v \leq' w \Leftrightarrow \exists u \in \Pi : uv \leq uw$
This comes from $\leq'$ on $\mathbb{Z}^n$!

**Example**
$\leq$ on the right
$\rightsquigarrow x^i y^j \leq' x^k y^l \Leftrightarrow i + j \leq k + l$




**Proposition**
$\exists$ nonempty ideal $T \subseteq \Pi$ that intersects each $\approx'$-class in a single $\approx$-class (an *asymptotic range*).

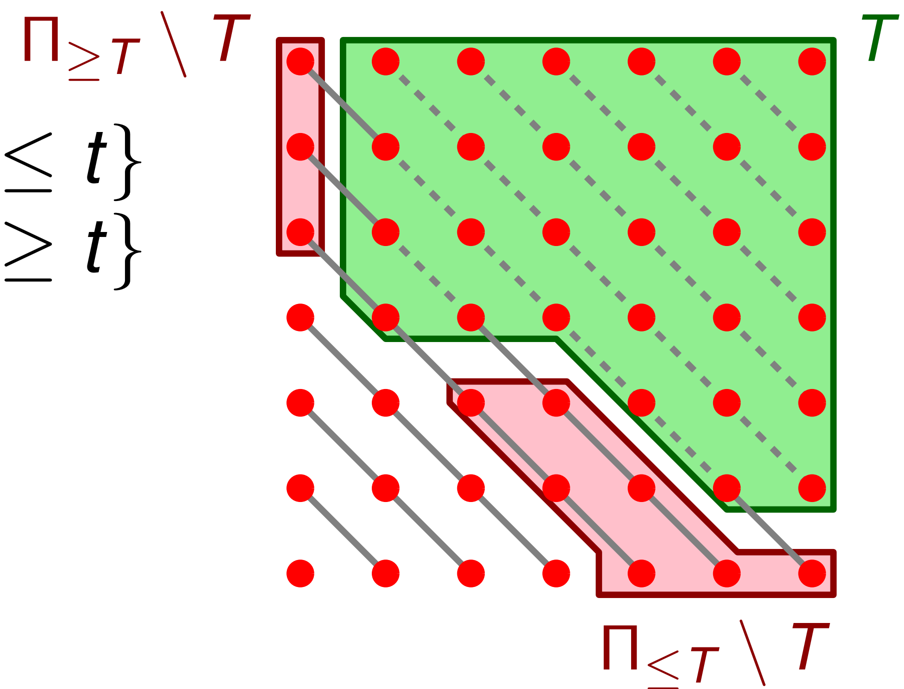
# 2b. Divide and conquer



**Two ideals containing $T$**

$\Pi_{\leq T} = \{u \in \Pi \mid \exists t \in T : u \approx' t, u \leq t\}$

$\Pi_{\geq T} = \{u \in \Pi \mid \exists t \in T : u \approx' t, u \geq t\}$




**Crucial idea**

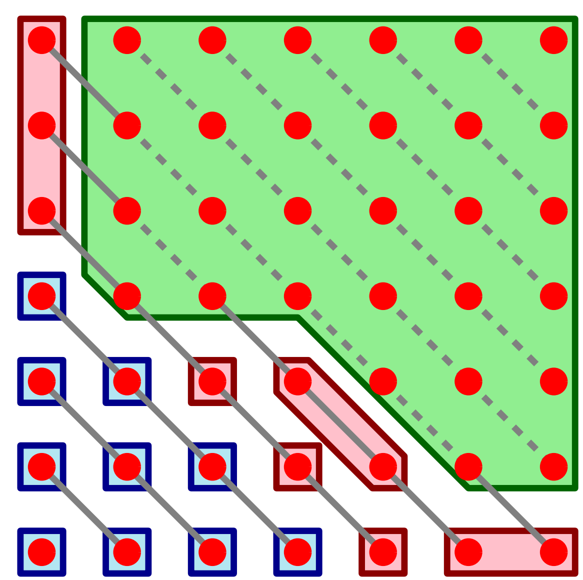
Partition $\Pi$ via $u \mapsto (\Pi_{\leq T} : u, \Pi_{\geq T} : u)$

$\rightsquigarrow$ parts $\neq T$ are f.g. $\mathrm{Mon}_{<n}$-sets

$\rightsquigarrow$ $\leq$ is determined by:

- $\leq'$ (preotree with numerical data)
- $\preceq$ on the (finite) set of parts
- the restrictions of $\leq$ to parts

$\rightsquigarrow$ recurse.



# Main results



**Theorem B**
If Π is finitely generated (a quotient of some $\mathrm{Mon}_n$), then $\mathcal{P}(\Pi)$ admits a countable cover by admissible sets.

**Theorem C**
The universal theory of ordered commutative monoids is decidable.

**Open problem**
A Chevalley theorem: given $\rho : \Pi_1 \to \Pi_2$ and a constructible set $C \subseteq \mathcal{P}(\Pi_2)$, is $\rho^*(C) \subseteq \mathcal{P}(\Pi_1)$ constructible?

**Thank you!**