Finiteness up to symmetry

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Part I: some questions

Moment varieties of mixtures

$Y = (Y_1, \dots, Y_n)$ $\mathbb{R}$-valued random variables $\rightsquigarrow$ *degree-d moments* given by $\mathbb{E}(Y^\alpha) = \mathbb{E}(\prod_i Y_i^{\alpha_i})$ where $\sum_i \alpha_i = d$.

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If $(X_{k1}, \dots, X_{kn})$ independent for $k = 1, \dots, r$ and $Y := X_k$ with probability p_k, then $\mathbb{E}(Y^\alpha) = \sum_{k=1}^r p_k \prod_i \mathbb{E}(X_{ki}^{\alpha_i})$.

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Question 1 [Alexandr-Kileel-Sturmfels, 2023]

Fix d and r, but allow n to vary. Is the *ideal of polynomial relations* among the $\binom{n+d-1}{d}$ dark red expressions generated by finitely many elements up to S_n?

Counting matrices

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Example
$S = \{0, 1\}$, $r = 2$

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & J & 0 \\ J^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} J_1 & J_2 & 0 \\ J_2^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\rightsquigarrow |M_n / S_n| = 2\lceil \tfrac{n}{2} \rceil \cdot \lfloor \tfrac{n}{2} \rfloor + \binom{n}{2}$$

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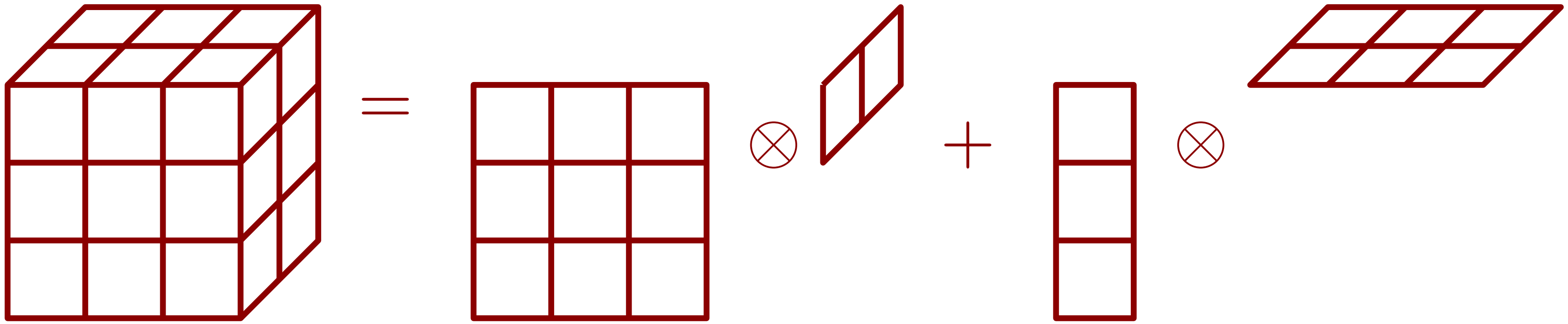

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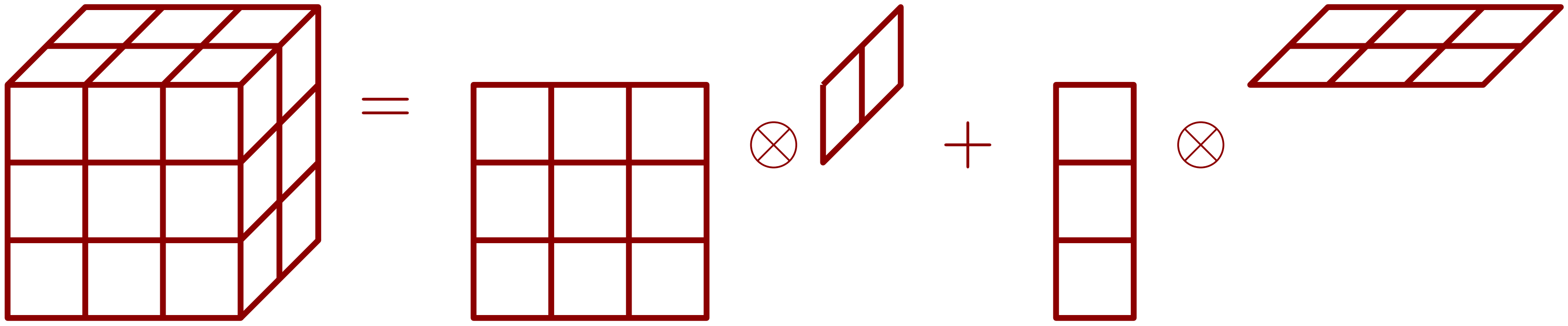

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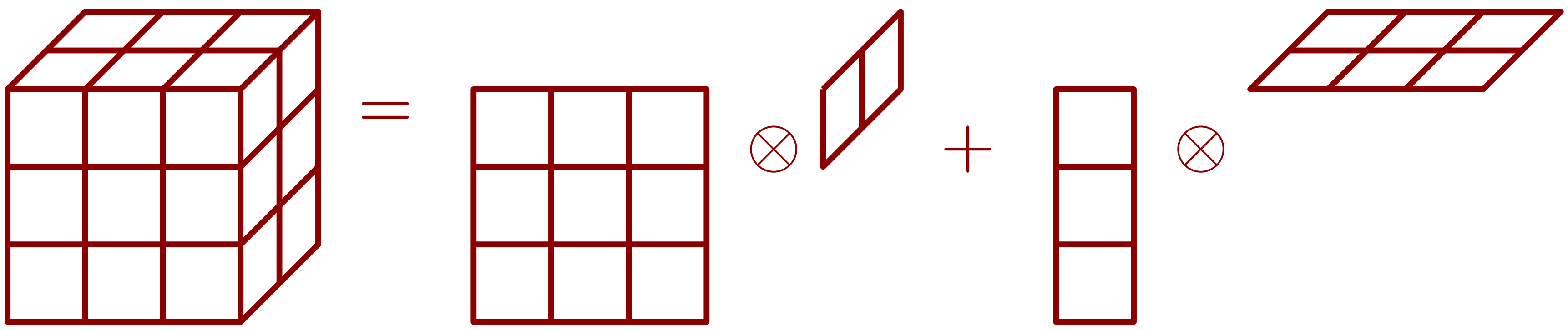

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Questions 3

Is there an efficient membership test for X?

For $K = \mathbb{C}$, is any $T \in \overline{X}$ equal to $\lim_{\epsilon \to 0} \sum_{i=1}^{r} S_i(\epsilon) \otimes T_i(\epsilon)$ for certain $S_i, T_i \in X(\mathbb{C}((\epsilon)))$ with bounded exponents of ϵ?

Part II: some theory, and some answers

FI-modules

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Example: $V(I) := \mathbb{C}I$, $V(\pi)\left(\sum_{i\in I} c_i i\right) = \sum_{i\in I} c_i \pi(i)$; and $W(I) := \{\sum_i c_i i \mid \sum_i c_i = 0\}$ defines an **FI**-submodule.

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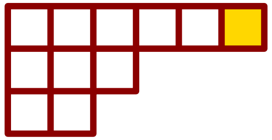
Theorem [Church-Ellenberg-Farb, 2015]
If V is a finitely generated **FI**-module, then
(a) for $n \gg 0$, the S_n-module $V([n])$ grows in a well-understood manner.



(b) every **FI**-submodule of V is also finitely generated.

FI-algebras

Theorem [D. Cohen, 1967,1987]

The **FI**-algebra $A : I \mapsto K[x_i \mid i \in I]$ determined by $A(\pi) : x_i \mapsto x_{\pi(i)}$ is *Noetherian:* any ideal in A is finitely generated.
The same holds for $A^{\otimes c}, c \in \mathbb{N}$.

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Theorem [Nagel-Römer, 2017]

Q an ideal in $A^{\otimes c}$ $\rightsquigarrow$ the bivariate Hilbert series $H_Q(s,t) = \sum_{n,d\geq 0} \dim[A^{\otimes c}([n])_{\leq d} / Q([n])_{\leq d}] s^n t^d$ is rational.

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Theorem [D-Eggermont-Farooq, 2022]
of S_n-orbits on $\{$minimal primes over $Q([n])\}$ is *quasipolynomial* for $n \gg 0$: $a_0(n) + \cdots + a_d(n)n^d$, a_i periodic.

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Corollary
In Question 2, $|M_n / S_n|$ is a quasipolynomial in n for $n \gg 0$.

Recall for $S = \{0, 1\}$, $r = 2$: $|M_n / S_n| = 2\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor + \binom{n}{2}$.

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Proof
Let $\mu : \mathbb{C}^{r\times n} \to \mathbb{C}^{n\times n}, M \mapsto M^T M$ be the multiplication map, and $X([n]) := \mu^{-1}(M_n)$. Note that μ is S_n-equivariant, and the irreducible components of $X([n])$ correspond bijectively to the minimal primes over the vanishing ideal $Q([n]) \subseteq A^{\otimes r}([n])$ of $X([n])$. Now use the theorem. $\square$

FI-algebras, continued

No Noetherianity for ≥ 2 indices: in $B(I) = K[y_{ij} \mid i, j \in I]$ the ideal $(y_{12} \cdot y_{21}, \quad y_{12} \cdot y_{23} \cdot y_{31}, \quad ...)$ is not f.g. But:

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Proof. Take $B(I) := K[y_\alpha | \alpha \in \{0, ..., d\}^I$ with $\sum_i \alpha_i = d]$ and $c := rd$. Variables in $A^{\otimes c}(I)$: x_{kei}, $k = 1, ..., r, e = 1, ..., d$, $i \in I$. Apply thm to $\varphi : y_\alpha \mapsto \sum_{k=1}^r \prod_{i, \alpha_i > 0} x_{k\alpha_i i}$. □

Polynomial functors

Write $\mathbf{Vec}_K$ for the category of *fin.-dim.* vector spaces.

Definition
A functor $P : \mathbf{Vec}_K \to \mathbf{Vec}_K$ is *polynomial* of degree $\leq d$ if $\forall U, V : P : \text{Hom}(U, V) \to \text{Hom}(P(U), P(V))$ is polynomial of degree $\leq d$.

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A *subset* $X \subseteq P$ is the data $(X(V) \subseteq P(V))_{V \in \mathbf{Vec}_K}$ such that $\forall V, W, \varphi : V \to W : P(\varphi)(X(V)) \subseteq X(W)$. It is *Zariski-closed* if each $X(V)$ is.

Example: Veronese and its secants

Take $K = \mathbb{C}$, $P(V) = S^d$, and $X(V) = \{\ell^d \mid \ell \in V\}$; so $X(\mathbb{C}^2) = \{ae_1^2 + be_1e_2 + ce_2^2 \mid b^2 - 4ac = 0\}$.

$X \subseteq P$ is closed, and its ideal is generated by $\Delta = b^2 - 4ac$: $\mathcal{I}(X(V)) = (\{P(\varphi)^\sharp(\Delta) \mid \varphi \in \mathrm{Hom}(V, \mathbb{C}^2)\})$.

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Define $\sigma_r X \subseteq P$ via $(\sigma_r X)(V) := \overline{\{\ell_1^d + \cdots + \ell_r^d \mid \ell_i \in V\}}$.

Remark: Closure is needed:
$\lim_{\epsilon \to 0}(\epsilon^{-1}e_1 + \epsilon^2 e_2)^3 + (-\epsilon^{-1}e_1 + \epsilon^2 e_2)^3 = 6e_1^2 e_2$.

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Proposition [Landsberg-Ottaviani 2013]
$\mathcal{I}(\sigma_r(X))$ is generated by $\mathcal{I}(\sigma_r(X))(\mathbb{C}^{r+1})$.

(Eqs for $(\sigma_r X)(\mathbb{C}^{r+1})$: finitely many, but not easy to find.)

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Theorem [D, 2019] and [D-Blatter-Rupniewski, 2023]
Let P be a polynomial functor and $X \subseteq P$ be a closed subset. Then there exists a $U \in \mathbf{Vec}$ such that for all $V \in \mathbf{Vec}$: $X(V) = \bigcap_{\varphi \in \mathrm{Hom}(V,U)} P(\varphi)^{-1}(X(U))$.

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Corollary: There exists n_0 such that for all $n \geq n_0$, $T \in P(K^n)$ lies in $X(K^n)$ iff $T|_I \in X(K^I)$ for all n_0-element subsets $I \subseteq [n]$. ($\rightsquigarrow$ X has a poly time membership test).

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Proof of Corollary: by the theorem, $\mathcal{I}(X)$ is the radical of some ideal generated in degree $\leq$ some d. Now $I \mapsto \{$polynomials on $P(K^I)$ of degree $\leq d\}$ is a finitely generated **FI**-module. Hence so is $\mathcal{I}(X)_{\leq d}$ by Church-Ellenberg-Farb. Take n_0 big enough to see all generators. $\square$

Theorem [Bik-D-Eggermont-Snowden, 2023]

Assume $K = \mathbb{C}$, let $\alpha : P \to Q$ be a *polynomial transformation*, i.e., $\alpha_V : P(V) \to Q(V)$ is a polynomial map and for all $\varphi : V \to W$ the following commutes:

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Then $\exists N$: for all V and all $q \in \overline{\mathrm{im}(\alpha_V)}$ there exists $p(\epsilon) \in P(V)(\mathbb{C}((\epsilon)))$ with exponents of ϵ all $\geq -N$ and $\lim_{\epsilon \to 0} \alpha_V(p(\epsilon)) = q$.

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$\rightsquigarrow$ positive answers to Questions 3 about partition rank!

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Thank you!