Tensors of bounded rank

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Bounded-rank tensors

Definition

Rank of $\omega \in V_0 \otimes \cdots \otimes V_{p-1}$ is minimal k in
$\omega = \sum_{i=0}^{k-1} v_{i0} \otimes \cdots \otimes v_{i,p-1}$

Border rank is minimal k such that
$\omega \in \overline{\{\text{rank} \leq k \text{ tensors}\}}$

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Border-rank $\leq k$ tensors are characterised by polynomial equations of bounded degree independent of p or the V_i

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Flattening and contracting:
$\flat : v_0 \otimes \cdots \otimes v_{p-1} \mapsto (v_0 \otimes \cdots \otimes v_{q-1}) \otimes (v_q \otimes \cdots \otimes v_{p-1})$
$v_0 \otimes \cdots \otimes v_{p-1} \mapsto x(v_{p-1}) \cdot v_0 \otimes \cdots \otimes v_{p-2}, \quad x \in V_{p-1}^*$
do not increase (b)rk

Reduction to single V

Lemma

$\phi_i : V_i \to U_i$ and $\omega \in V_0 \otimes \cdots \otimes V_{p-1}$
$\rightsquigarrow \operatorname{brk}(\phi_0 \otimes \cdots \otimes \phi_{p-1})\omega \leq \operatorname{brk}\omega$

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Lemma

$\omega \in V_0 \otimes \cdots \otimes V_{p-1}$ of brk $> k$

$\rightsquigarrow \exists\, \phi_i : V_i \to K^{k+1} =: V$ with $\operatorname{brk}(\phi_0 \otimes \cdots \otimes \phi_{p-1})\omega > k$

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$\omega : \bigotimes_{i \neq j} V_i^* \longrightarrow V_i$, image=: U_i

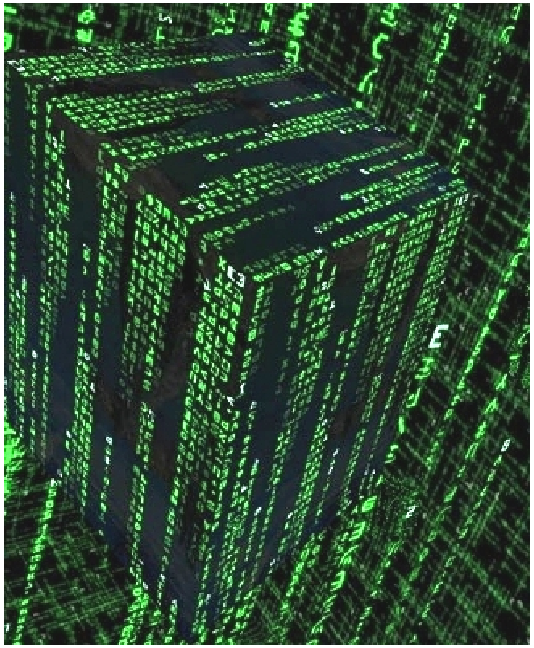
- if $\dim U_i \leq k$ choose $\phi : V_i \to V, \psi_i : V \to V_i$ such that $\psi_i \circ \phi_i$ is 1 on U_i
- if $\dim U_i > k$ take ϕ_i restricting to surjection $U_i \to V$

$(\psi_0 \otimes \cdots \otimes \psi_{p-1})(\phi_0 \otimes \cdots \otimes \phi_{p-1})\omega = \omega$ or $\mathrm{rk}\flat_{i,[p]-i}\omega > k$

□

Infinite-dimensional tensors, an impression

A wrong-titled movie. . .



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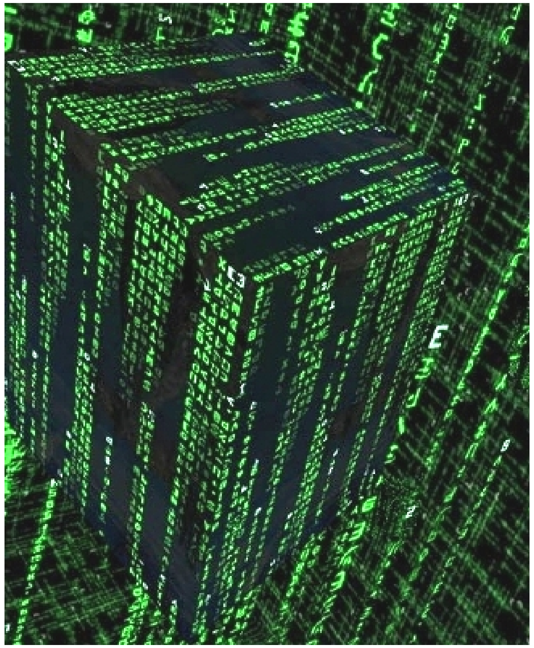
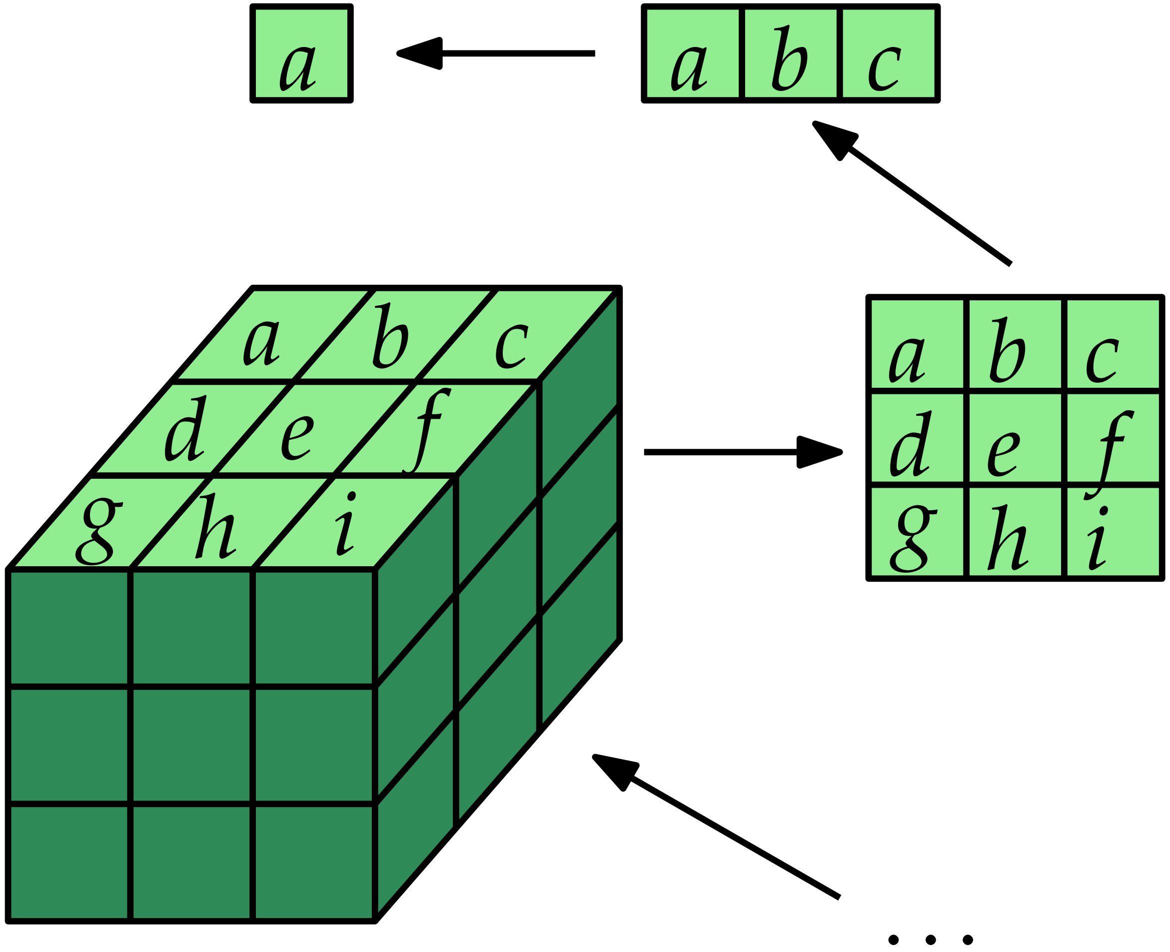
. . . and an infinite-dimensional tensor





Infinite-dimensional tensors

$x_0, \ldots, x_{n-1}$ basis of V^*

$$\begin{tikzcd}
V^{\otimes 0} & V^{\otimes 1} \arrow[l, "\langle\,\cdot\,, x_0\rangle"'] & V^{\otimes 2} \arrow[l] & \ldots \arrow[l] & V^{\otimes \infty} \arrow[l]
\end{tikzcd}$$

$V^{\otimes \infty}$ is dual space of U (but not vice versa)

$$\begin{tikzcd}
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basis of U: $x_w, w \in [n]^{\mathbb{N}}$, finitely many non-zero entries
$SU = K[x_w]$ coordinate ring of $V^{\otimes \infty}$

Example (minor of a flattening)
$w_1 = 10 \cdots, w_2 = 20 \cdots, w_1' = 0 \cdots, w_2' = 030 \cdots$

$$x[(w_1, w_2), (w_1', w_2')] = \begin{bmatrix} x_1 & x_{13} \\ x_2 & x_{23} \end{bmatrix} \quad (\operatorname{supp}(w_i) \cap \operatorname{supp}(w_i') = \emptyset)$$

Bounded-rank tensors, proof outline

$X_p \subseteq V^{\otimes p}$ border rank $\leq k$
$Y_p \subseteq V^{\otimes p}$ all flattenings have rank $\leq k$

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Noetherian up to symmetry!

$G_\infty = \bigcup_{n\in\mathbb{N}} \mathrm{Sym}(p) \ltimes \mathrm{GL}(V)^p$ acts on $V^{\otimes\infty}, Y_\infty, X_\infty$

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Subgoal

Y_∞ defined by finitely many G_∞-orbits of minors

The flattening variety I

Proposition

$W \subseteq V^{\otimes p}$ subspace, $k < p \rightsquigarrow$ if $\dim W > k$
then $\exists x \in V^*, i \in [p]$: $\dim\langle W, x\rangle_i > k$

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otherwise $\exists$ counterexample W spanned by
vectors $e_{i_0} \otimes \cdots \otimes e_{i_{p-1}},\ i_0, \ldots, i_{p-1} \in [n]$ (Borel fixpoint thm)

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$W \rightsquigarrow k + 1$ distinct words $u_0, \ldots, u_k \in [n]^p$

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$W \rightsquigarrow k + 1$ distinct words $u_0, \ldots, u_k \in [n]^p$

$\exists i \in [p] : u_0, \ldots, u_k$ *remain distinct when deleting i-th letter*

corresponding basis vectors remain linearly independent
when contracting with $\sum x_j$ at position i $\square$

Corollary
$Y_\infty^{\leq k}$ is defined by finitely many orbits of minors

The flattening variety II

Theorem
$\forall k \; Y_\infty^{\leq k} \subseteq V^{\otimes \infty}$ is G_∞-Noetherian

induction on k: $Y_\infty^{\leq 0} = \{0\}$ is, assume true for $k - 1$

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$Y_\infty^{\leq k-1}$ defined by G_∞ orbits of $\det_0, \ldots, \det_{N-1}$
$U_a := \{y \in Y_\infty^{\leq k} \mid \det_a \neq 0\}$
$Y_\infty^{\leq k} = Y_\infty^{\leq k-1} \cup G_\infty U_0 \cup \ldots \cup G_\infty U_{N-1}$

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Claim
each $G_\infty U_i$ is G_∞-Noetherian

$\rightsquigarrow$ both theorems

The flattening variety III

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each $G_\infty U_i$ is G_∞-Noetherian

Example

$k = 2$ and U open set where det $\begin{array}{c|cc} & 00 & 03 \\ \hline 10 & x_1 & x_{13} \\ 20 & x_2 & x_{23} \end{array} \neq 0$

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on $U \subseteq Y_\infty^{\leq 3}$ all x_w are expressible in terms of $x_{w'}$ with $|\mathrm{supp}(w') \setminus \{0,1\}| \leq 1$

e.g. $w = 2113 = 2010 + 0103 \rightsquigarrow \det \begin{array}{c|ccc} & 0000 & 0300 & 0103 \\ \hline 1000 & x_1 & x_{13} & x_{1103} \\ 2000 & x_2 & x_{23} & x_{2103} \\ 2010 & x_{201} & x_{231} & x_{2113} \end{array}$

$= dx_{2113} +$ smaller support $= 0 \rightsquigarrow x_{2113} = (\text{smaller supp})/d$

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Fundamental theorem $\rightsquigarrow$ U $\mathrm{Sym}(\{2, 3, \ldots\})$-Noetherian
$\rightsquigarrow$ $G_\infty U$ is G_∞-Noetherian $\square$

Alternating tensors

$$V = \langle \ldots, \mathbf{3/2}, \mathbf{1/2}, \mathbf{-1/2}, \mathbf{-3/2}, \ldots \rangle$$

$$V_{p,n} := \langle \mathbf{p - 1/2}, \ldots, \mathbf{-n + 1/2} \rangle \subseteq V$$

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$\bigwedge^{\infty/2} V := \lim_{\to p,n} \bigwedge^n V_{pn}$ *infinite wedge*

$(\bigwedge^{\infty/2} V)^* \supseteq$ inverse Grassmannian X

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Conjecture

the k-th secant variety of X is defined in bounded degree

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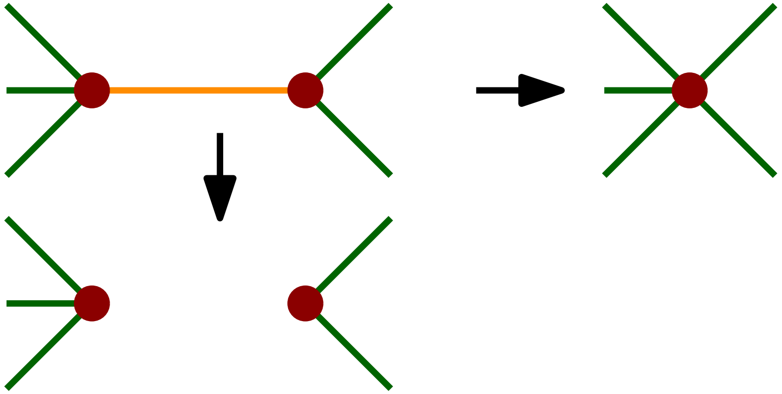
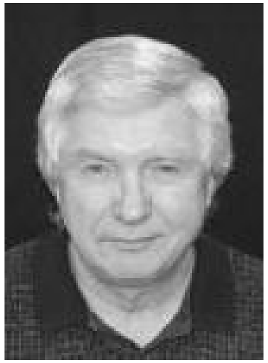
Conjecture HARD!

X is (set-theoretically) $\mathrm{Sym}(\mathbb{Z} + 1/2)$-Noetherian

Matroid minor conjecture

Robertson-Seymour
Graphs well-partially-ordered by





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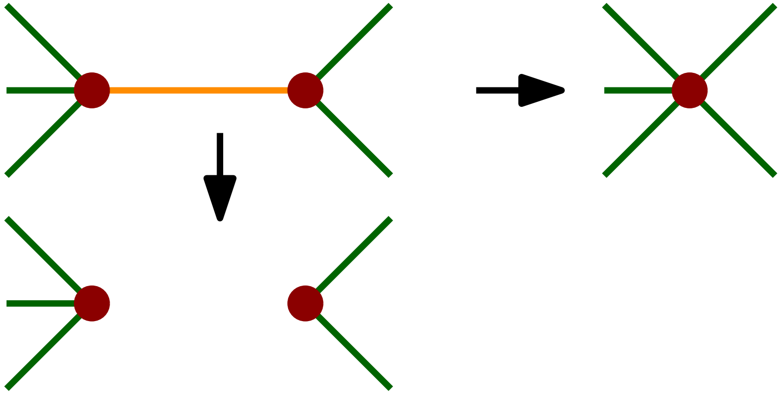

Geelen-Gerards-Whittle

$\mathbb{F}_2$-matroids well-partially ordered!

Other finite fields?

Matroid minor conjecture

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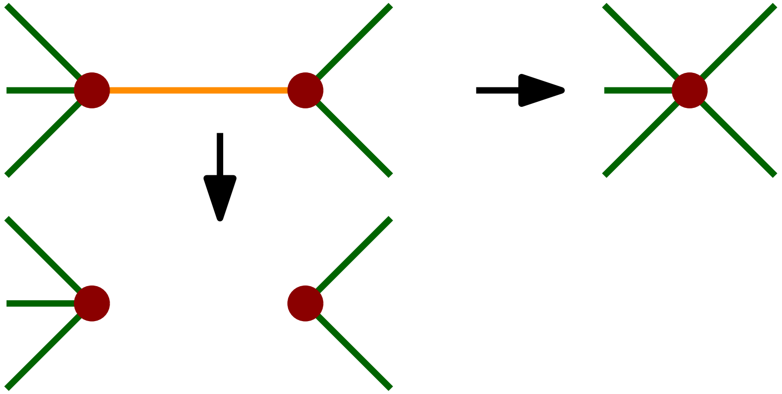

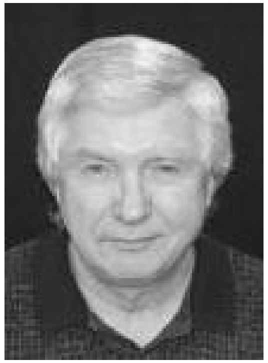

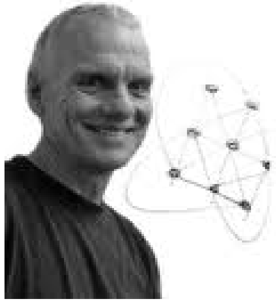

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Conjecture

Inverse Grassmannian (set-theoretically)

Sym($\mathbb{Z} + 1/2$)-Noetherian makes sense for infinite fields!