Finite up to symmetry

Jan Draisma
TU Eindhoven

Berkeley Colloquium/RTG Workshop, September 2012

Hilbert's Basis Theorem

David Hilbert (1862-1943)

$$f_1(x_1, \ldots, x_n) = 0$$
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Bruno Buchberger

Gröbner bases, algorithmic methods

(Non-)Noetherian rings

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$x_1 = 0, x_2 = 0, x_3 = 0, \ldots$

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R *Noetherian* if

$f_1, f_2, \ldots \in R$

$\Rightarrow \exists j$ with $f_j \in Rf_1 + Rf_2 + \ldots + Rf_{j-1}$

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$K[x_1, x_2, \ldots, x_n]$ Noetherian,
$K[x_1, x_2, \ldots]$ not Noetherian,
…but it *is up to symmetry!*

Fundamental Theorem

$$R := K\begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ \vdots & \vdots & \vdots & \\ x_{k-1,0} & x_{k-1,1} & x_{k-1,2} & \cdots \end{bmatrix}$$

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Cohen, Aschenbrenner/Hillar/Sullivant

$f_0, f_1, \ldots \in R \rightsquigarrow \exists i : \forall j \geq i :$

$f_j \in R \cdot \mathrm{Inc}(\mathbb{N}) f_0 + \ldots + R \cdot \mathrm{Inc}(\mathbb{N}) f_{i-1}$

(R *Inc*($\mathbb{N}$)-*Noetherian* $\rightsquigarrow$ Sym($\mathbb{N}$)-Noetherian)

...ist das Theologie?

Equivariant Buchberger algorithm
X infinite set of variables
Π monoid acting on X
$\prec$ well-order on monomials with
$v \prec w \Rightarrow uv \prec uw, \pi v \prec \pi w$

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Definition
$I \subseteq K[X]$ a Π-stable ideal
$B \subseteq I$ is a Π-*Gröbner basis*
if $\forall f \in I : \exists b \in B, \pi \in \Pi : \mathrm{lm}(\pi b) | \mathrm{lm} f$

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Theorem (many people)
$\exists$ a Π-equivariant Buchberger algorithm computing a Π-Gröbner basis, *provided that it terminates*
(and it does for $K[x_{ij} \mid i \in [k], j \in \mathbb{N}], \Pi = \mathrm{Inc}(\mathbb{N})$)

I: Second hypersimplex

Example

variables $y_{ij}, x_i (i, j \in \mathbb{N}, i > j)$

$K[y_{ij}] \longrightarrow K[x_i], y_{ij} \mapsto x_i x_j$

Inc($\mathbb{N}$)-equivariant; *kernel?*

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generated by 2×2-minors of symmetric matrix y not containing diagonal entries

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Output
$x_0 x_1 - y_{10},\ x_2 y_{10} - x_1 y_{20},\ x_2 y_{10} - x_0 y_{21},\ x_1 y_{20} - x_0 y_{21}$
$x_0^2 y_{21} - y_{20} y_{10},\ y_{32} y_{10} - y_{30} y_{21},\ y_{31} y_{20} - y_{30} y_{21}$

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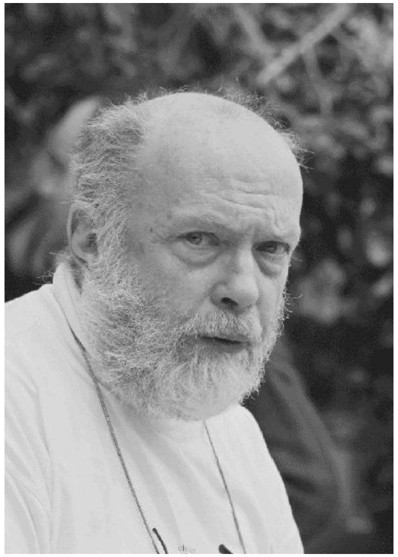
II: Vandermonde relations



Andreas Dress

$y_0, y_1, \ldots$ variables

$z_I := \prod_{i,j\in I, i<j}(y_i - y_j)$ for $|I| = k$, fixed

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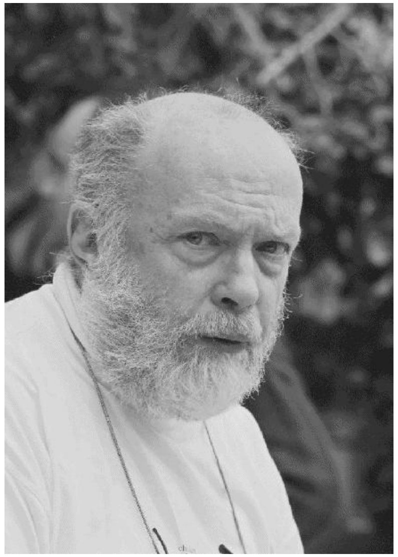

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Theorem

Yes (at least in characteristic zero).

$$z_I = \det \begin{bmatrix} 1 & \cdots & 1 \\ y_{i_0} & \cdots & y_{i_{k-1}} \\ \vdots & & \vdots \\ y_{i_0}^{k-1} & \cdots & y_{i_{k-1}}^{k-1} \end{bmatrix}$$ satisfy *Plücker relations*.

mod these $\rightsquigarrow$

invariant ring $K[x_{ij}, i \in [n], j \in \mathbb{N}]^{\mathrm{SL}_n}$ is $\mathrm{Inc}(\mathbb{N})$-Noetherian $\square$

Noetherianity for topological spaces

Definition
Π monoid acting on topological space X
$\rightsquigarrow$ X is Π-*Noetherian* if every chain $X_0 \supseteq X_1 \supseteq \ldots$
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Lemma
R a Π-Noetherian K-algebra $\rightsquigarrow$ $\{K\text{-valued points of } R\}$
Π-Noetherian space ($\rightsquigarrow$ $K^{k\times\mathbb{N}}$ is $\mathrm{Inc}(\mathbb{N})$-Noetherian)

III: Bounded-rank tensors

Definition

Rank of $\omega \in V_0 \otimes \cdots \otimes V_{p-1}$ is minimal k in
$\omega = \sum_{i=0}^{k-1} v_{i0} \otimes \cdots \otimes v_{i,p-1}$

Border rank is minimal k such that
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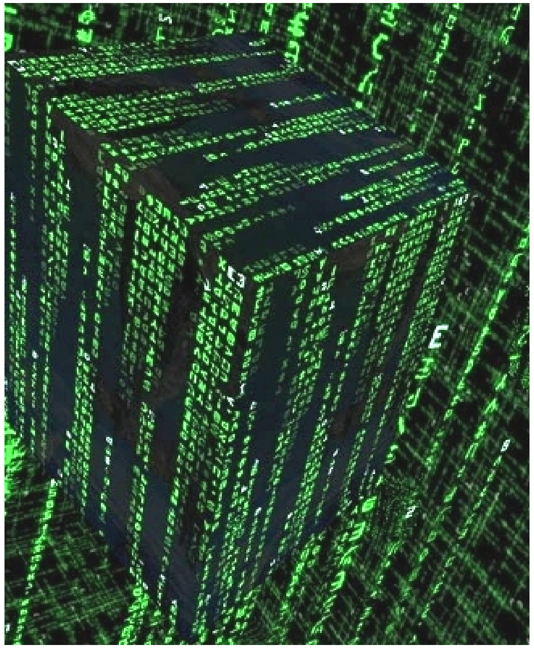
Flattening and contracting:

$$v_0 \otimes \cdots \otimes v_{p-1} \mapsto (v_0 \otimes \cdots \otimes v_{q-1}) \otimes (v_q \otimes \cdots \otimes v_{p-1})$$

$$v_0 \otimes \cdots \otimes v_{p-1} \mapsto x(v_{p-1}) \cdot v_0 \otimes \cdots \otimes v_{p-2}, \quad x \in V_{p-1}^*$$

Intermezzo: infinite-dimensional tensors

A wrong-titled movie. . .



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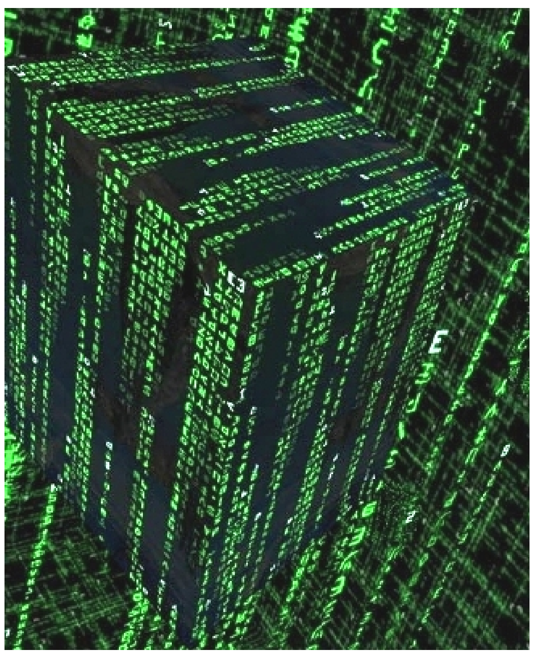
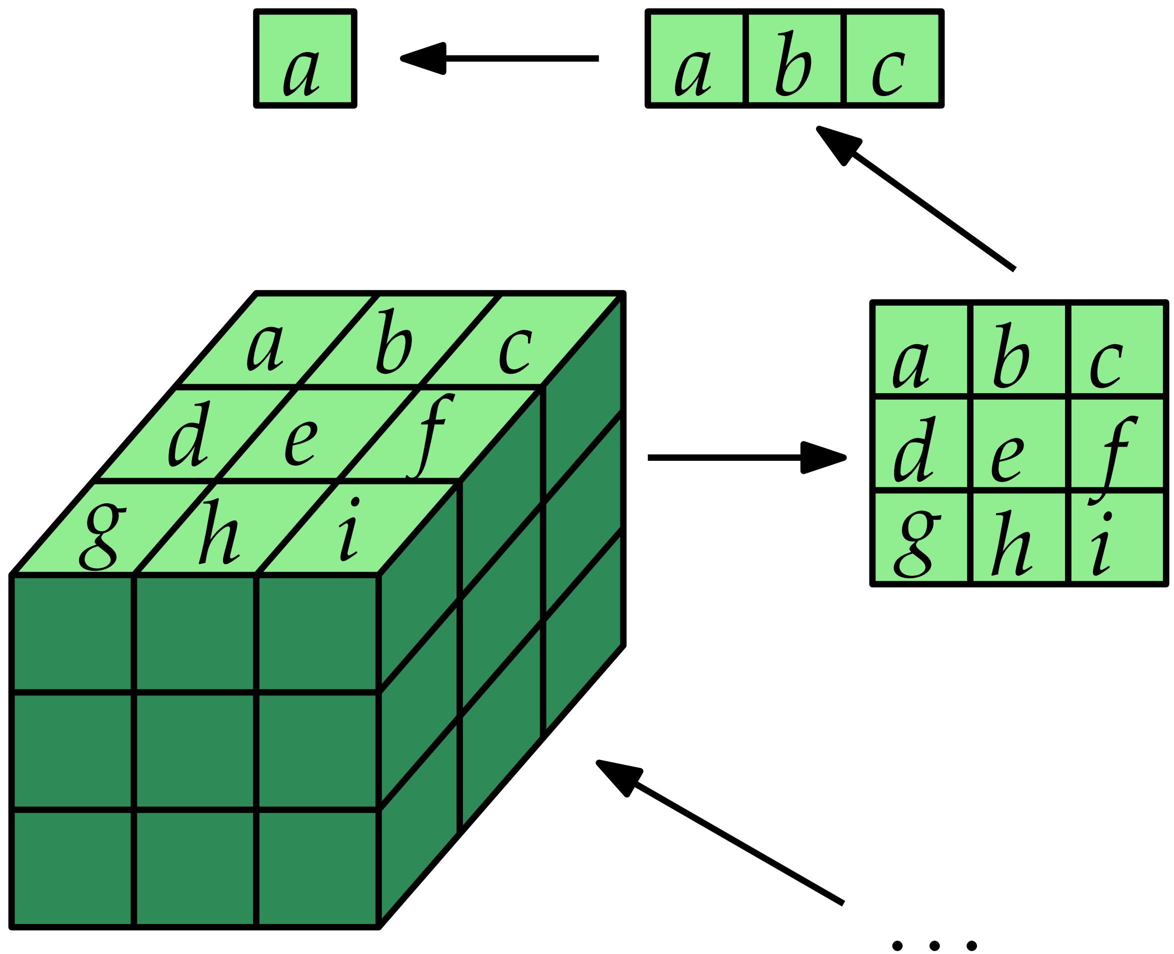
…and an infinite-dimensional tensor





Bounded-rank tensors, proof sketch

$X_p \subseteq V^{\otimes p}$ border rank $\leq k$
$Y_p \subseteq V^{\otimes p}$ all flattenings have rank $\leq k$

$x_0 \in V^*$

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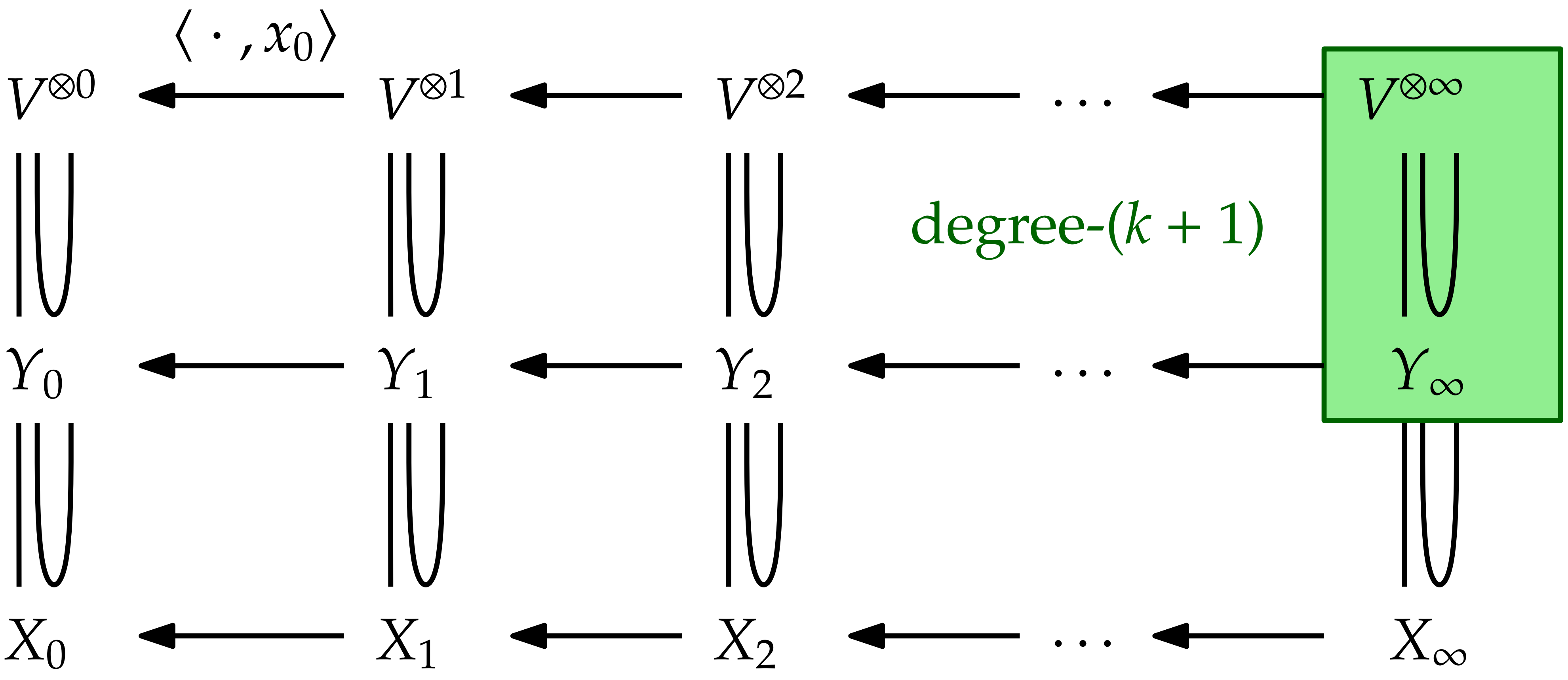

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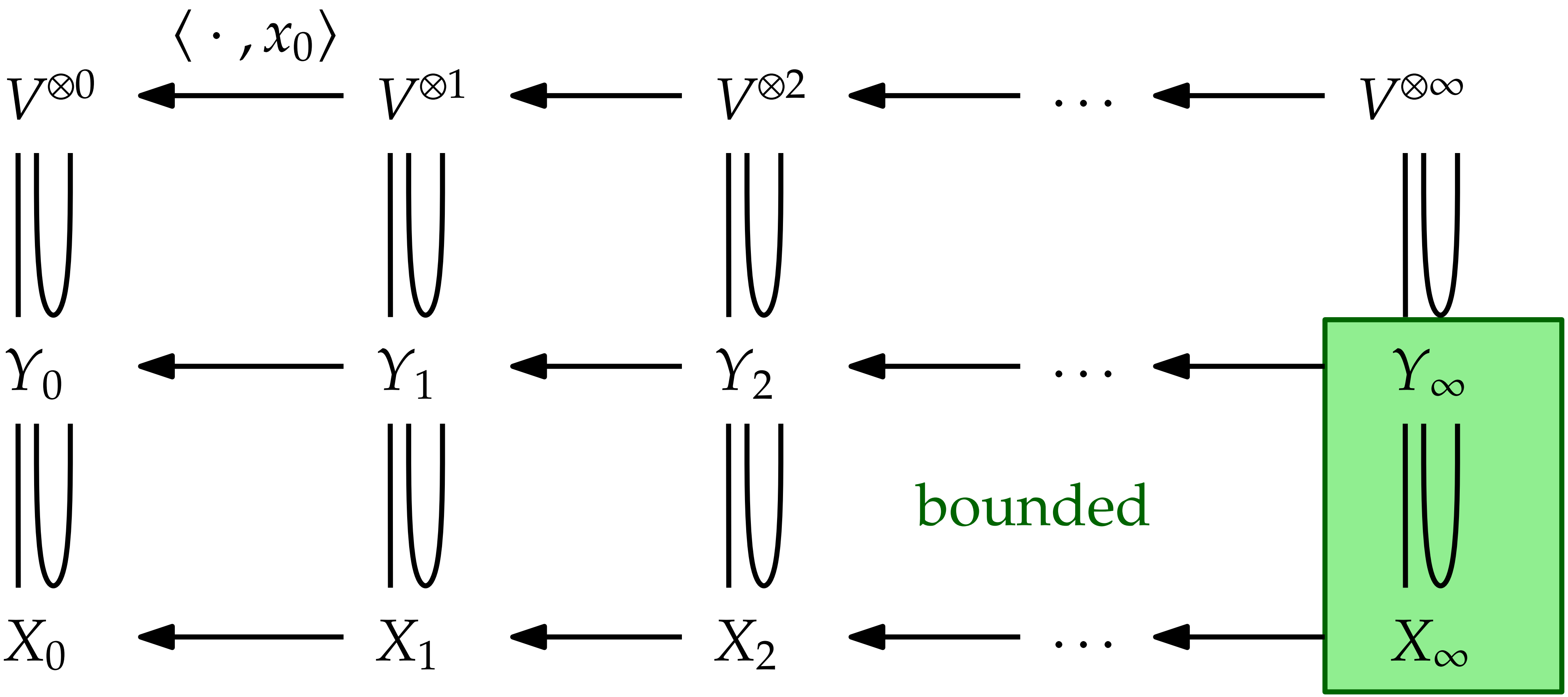

Y_∞ Noetherian (artist's impression)

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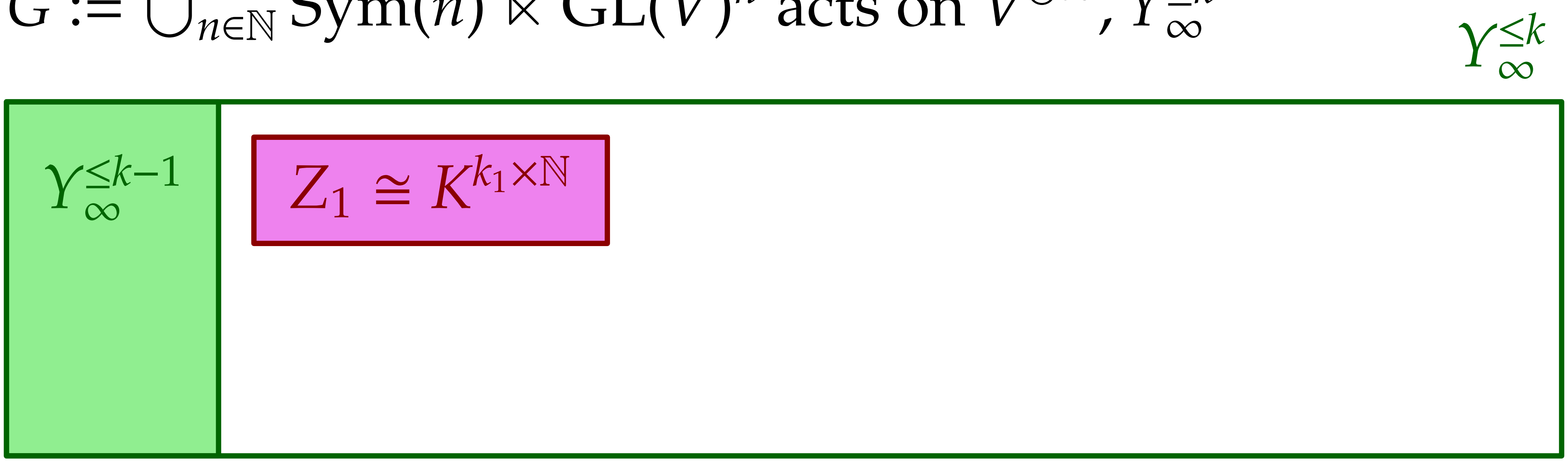

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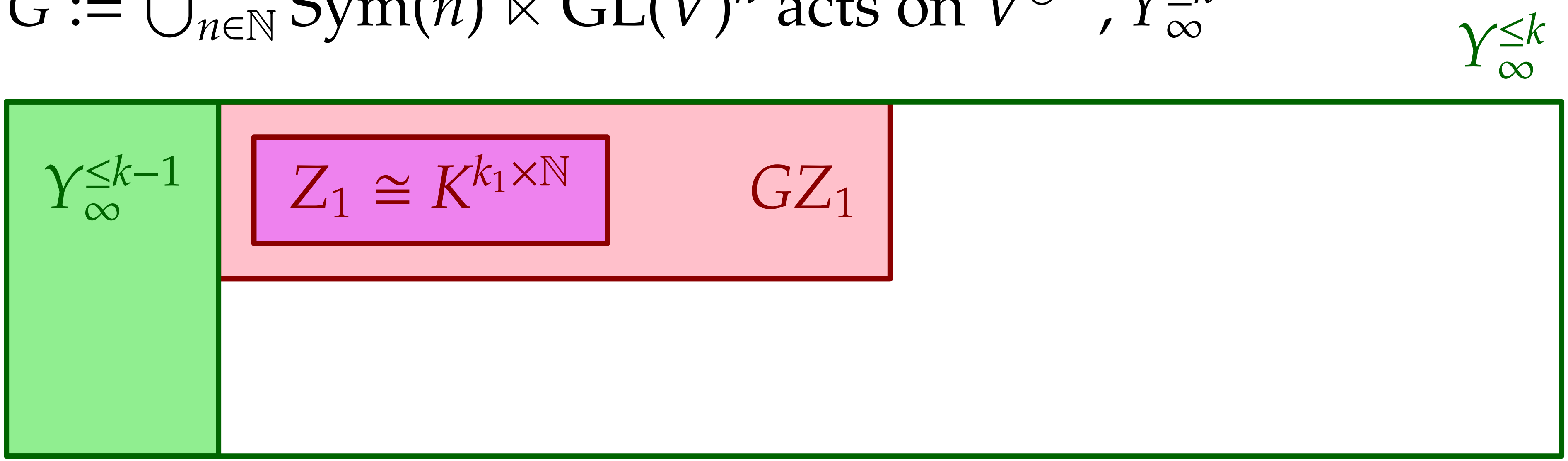

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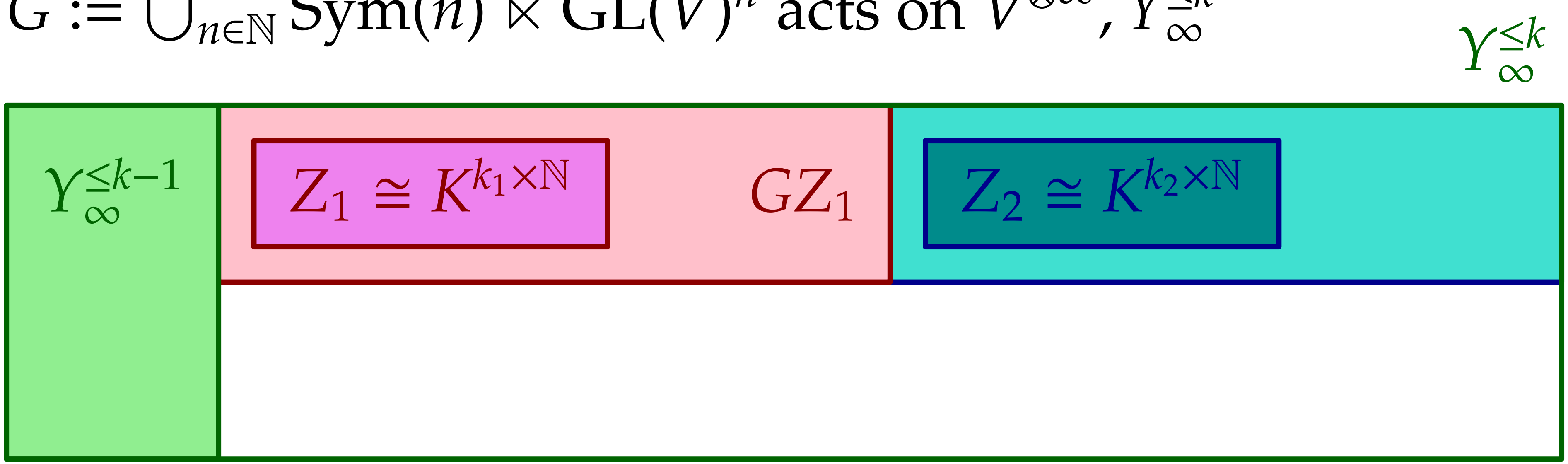

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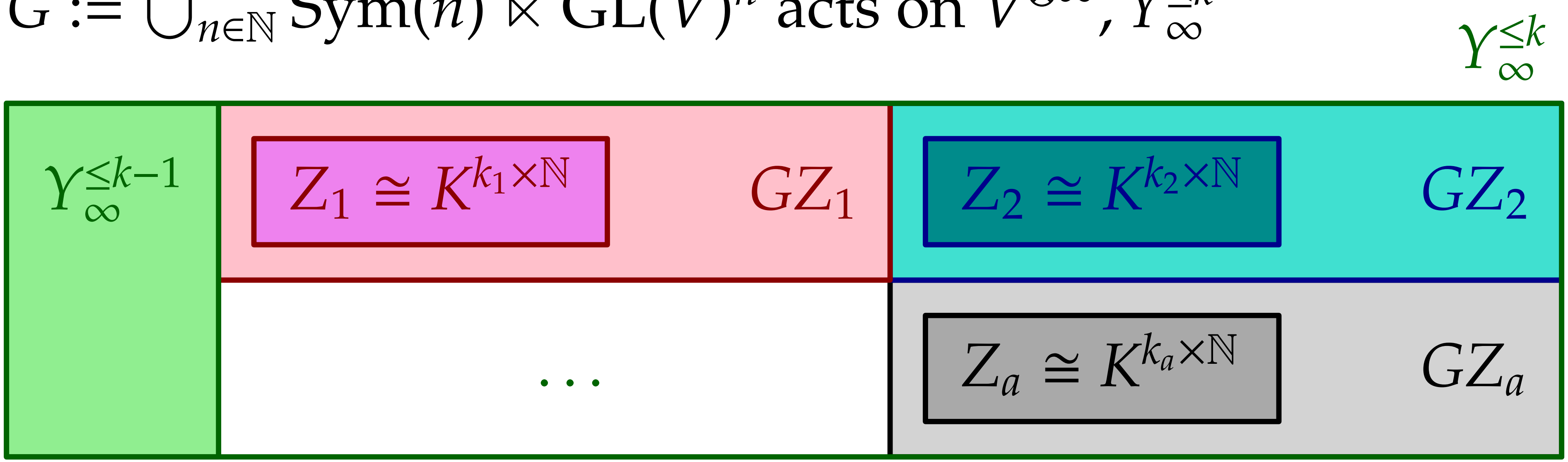

Further topics

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defined in bounded degree, independent of the tree (with Rob Eggermont)

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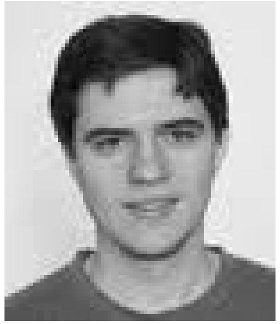
Syzygies of Segre (Andrew Snowden)
For each k, the Segre Embedding
$\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_p \to \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$
has finitely many types of syzygies



Further topics

Phylogenetic tree models
defined in bounded degree, independent of the tree (with Rob Eggermont)

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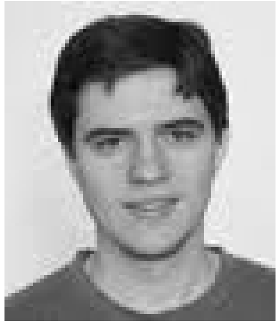
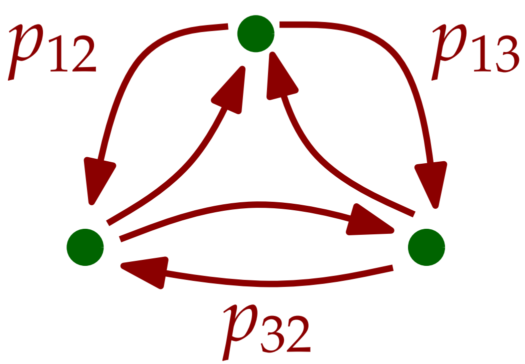


Markov chains (Ruriko Yoshida et al)



$\text{Prob}(1232) = p_{12}p_{23}p_{32}$

Relations among path probabilities stabilise as $T \to \infty$?

Further topics?

Skew-symmetric tensors
Inverse Grassmannian
Infinite wedge
Matroid minor conjecture

Symmetric tensors
Inverse Veronese
Infinite symmetric power

Computational issues

...

Further topics?

Skew-symmetric tensors
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Computational issues

. . .

Many infinite-dimensional varieties can be described with finitely many equations up to symmetry.