Well-partial orders, equivariant Gröbner Bases, and applications

Jan Draisma
TU Eindhoven

Berkeley, RTG workshop, September 2012

Well-partial orders

Definition

$\leq$ on S is *well-partial-order* if

$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

Well-partial orders

Definition
$\leq$ on S is *well-partial-order* if
$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

$\rightsquigarrow$ in fact $\exists i_0 < i_1 < \ldots : s_{i_0} \leq s_{i_1} \leq \ldots$

Well-partial orders

Definition
$\leq$ on S is *well-partial-order* if
$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

$\leadsto$ in fact $\exists i_0 < i_1 < \ldots : s_{i_0} \leq s_{i_1} \leq \ldots$

Lemma
S, T well-partially-ordered $\leadsto$ so is $S \times T$ with
$(s, t) \leq (s', t') :\Leftrightarrow s \leq s'$ and $t \leq t'$

Well-partial orders

Definition

$\leq$ on S is *well-partial-order* if

$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

$\leadsto$ in fact $\exists i_0 < i_1 < \ldots : s_{i_0} \leq s_{i_1} \leq \ldots$

Lemma

S, T well-partially-ordered $\leadsto$ so is $S \times T$ with

$(s, t) \leq (s', t') :\Leftrightarrow s \leq s'$ and $t \leq t'$

$(s_0, t_0) \quad (s_1, t_1) \quad (s_2, t_2) \quad (s_3, t_3) \quad (s_4, t_4) \quad (s_5, t_5) \quad \ldots$

Well-partial orders

Definition

$\leq$ on S is *well-partial-order* if

$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

$\rightsquigarrow$ in fact $\exists i_0 < i_1 < \ldots : s_{i_0} \leq s_{i_1} \leq \ldots$

Lemma

S, T well-partially-ordered $\rightsquigarrow$ so is $S \times T$ with

$(s, t) \leq (s', t') :\Leftrightarrow s \leq s'$ and $t \leq t'$

$$\begin{array}{ccccccc}
(s_0, t_0) & (s_1, t_1) & (s_2, t_2) & (s_3, t_3) & (s_4, t_4) & (s_5, t_5) & \ldots \\
\rightsquigarrow & (s_1, t_1) & & (s_3, t_3) & (s_4, t_4) & \ldots & \\
 & s_1 & \leq & s_3 \quad \leq & s_4 \quad \leq & \ldots &
\end{array}$$

Well-partial orders

Definition

$\leq$ on S is *well-partial-order* if

$s_0, s_1, s_2, \ldots \in S \Rightarrow \exists i < j : s_i \leq s_j$

$\rightsquigarrow$ in fact $\exists i_0 < i_1 < \ldots : s_{i_0} \leq s_{i_1} \leq \ldots$

Lemma

S, T well-partially-ordered $\rightsquigarrow$ so is $S \times T$ with

$(s, t) \leq (s', t') :\Leftrightarrow s \leq s'$ and $t \leq t'$

$$
\begin{array}{ccccccc}
(s_0, t_0) & (s_1, t_1) & (s_2, t_2) & (s_3, t_3) & (s_4, t_4) & (s_5, t_5) & \ldots \\
\rightsquigarrow & (s_1, t_1) & & (s_3, t_3) & (s_4, t_4) & \ldots & \\
& s_1 & \leq & s_3 & \leq \; s_4 & \leq \; \ldots & \\
\rightsquigarrow & & & t_i & \leq & t_j & \square
\end{array}
$$

Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

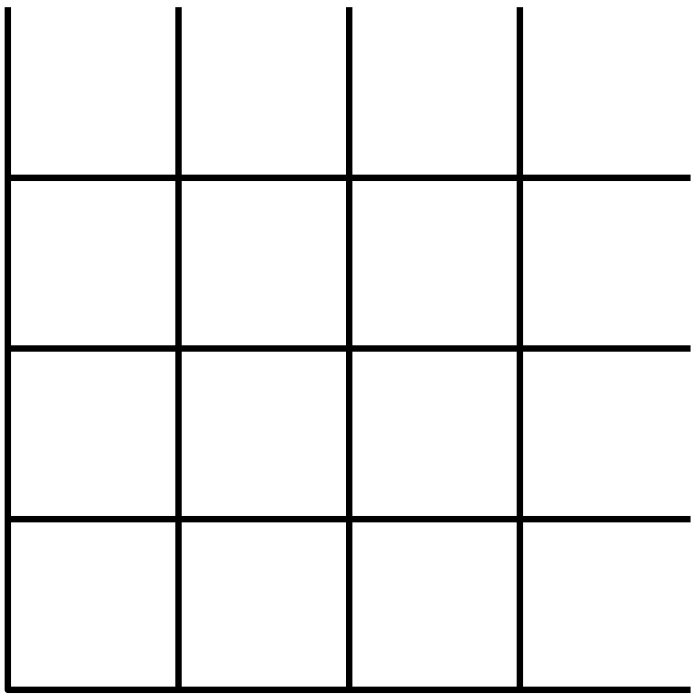
($\mathbb{N}^n$ is well-partially-ordered)

Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

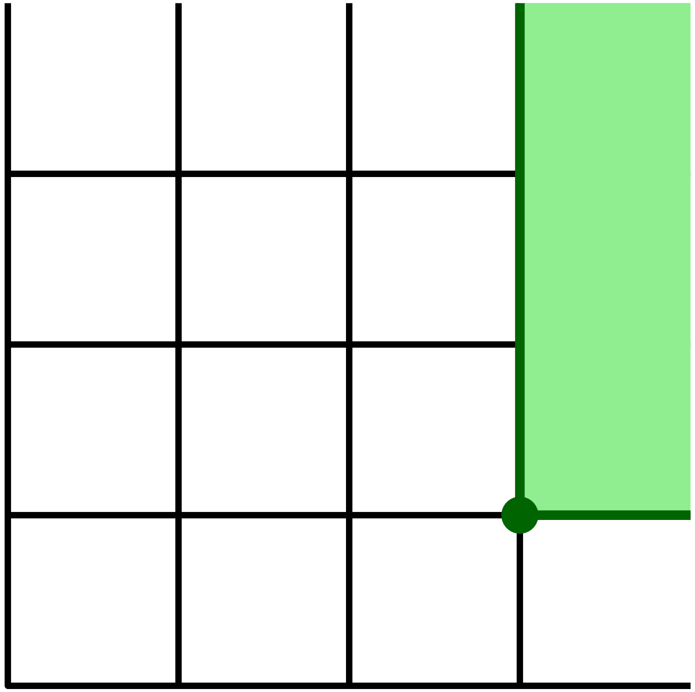
($\mathbb{N}^n$ is well-partially-ordered)



Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$
($\mathbb{N}^n$ is well-partially-ordered)

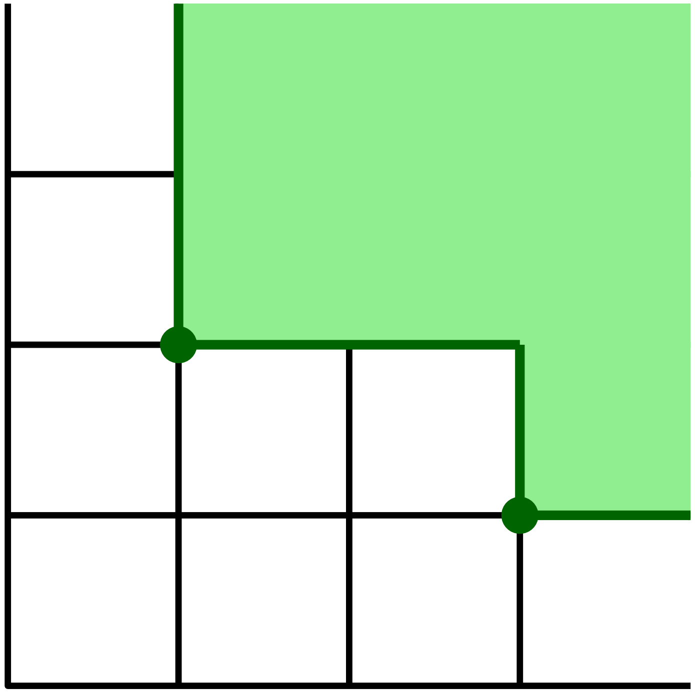


Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

($\mathbb{N}^n$ is well-partially-ordered)

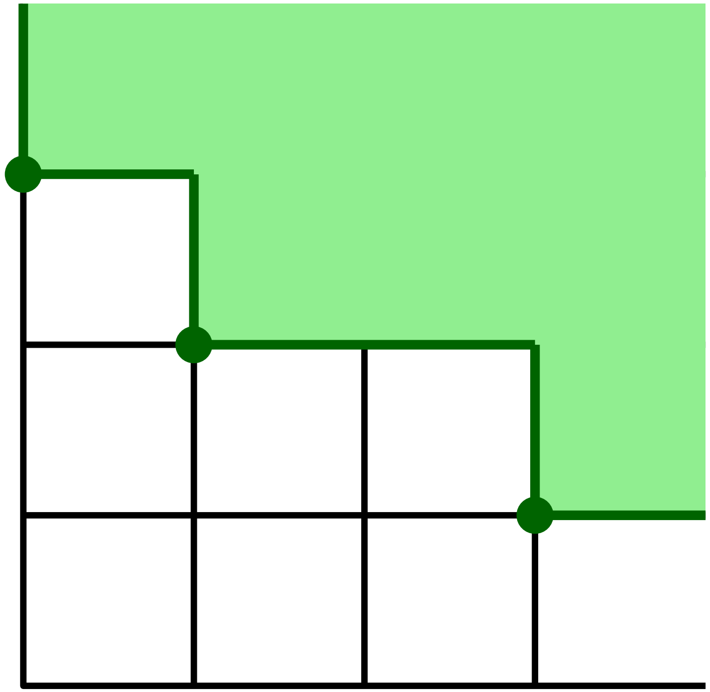


Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

($\mathbb{N}^n$ is well-partially-ordered)

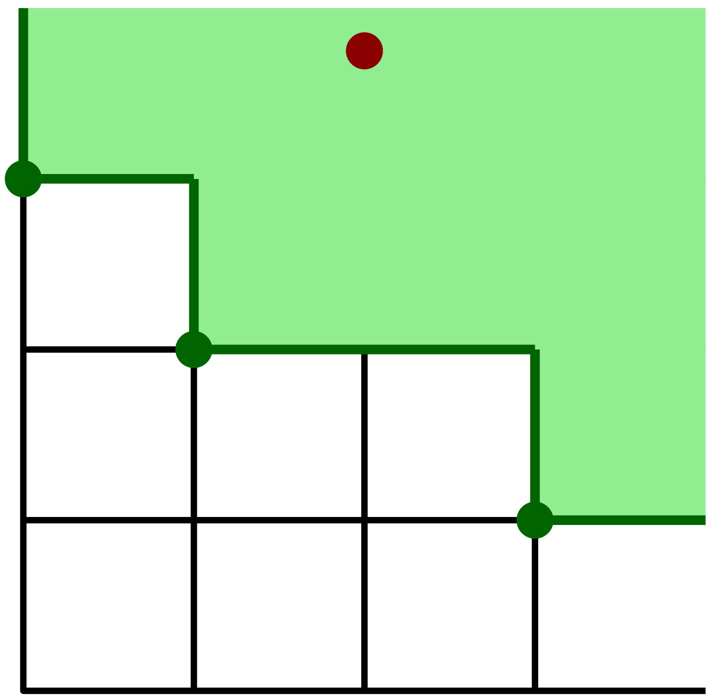


Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

($\mathbb{N}^n$ is well-partially-ordered)

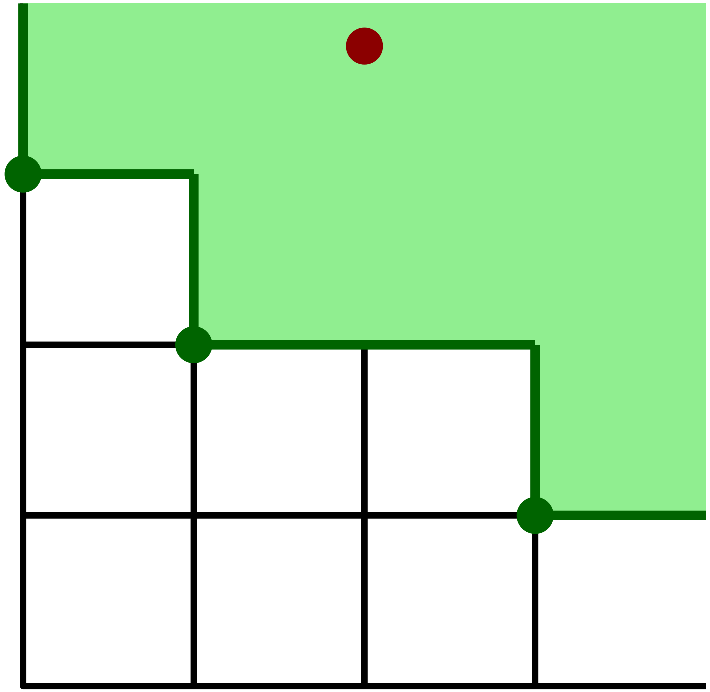


Dixon's Lemma

Leonard E. Dixon (1874-1954)

$\alpha_0, \alpha_1, \ldots \in \mathbb{N}^n \Rightarrow \exists i < j$ with $\alpha_j - \alpha_i \in \mathbb{N}^n$

($\mathbb{N}^n$ is well-partially-ordered)



well-quasi-order $\Leftrightarrow$ *upsets* finitely generated

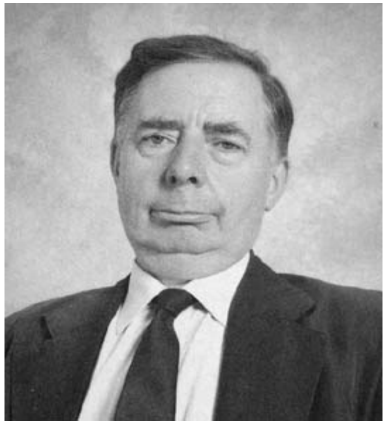
Higman's Lemma

Graham Higman (1917-2008)
$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$

Higman's Lemma

Graham Higman (1917-2008)
$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$

Crispin Nash-Williams (1932–2001)



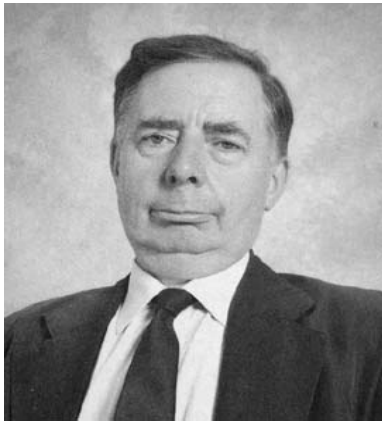
Higman's Lemma

Graham Higman (1917-2008)
$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$

Crispin Nash-Williams (1932–2001)

bad sequence of words, minimal lengths

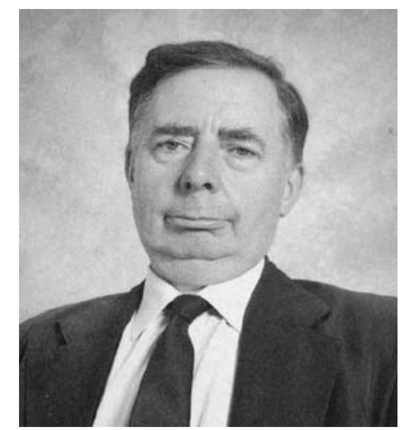
$w_0 \quad w_1 \quad w_2 \quad w_3 \quad w_4$



Higman's Lemma

Graham Higman (1917-2008)

$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$



Crispin Nash-Williams (1932–2001)

bad sequence of words, minimal lengths

$$\begin{array}{ccccccc} w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & \ldots \\ || & || & || & || & || & || & \\ s_0 v_0 & s_1 v_1 & s_2 v_2 & s_3 v_3 & s_4 v_4 & s_5 v_5 & \ldots \end{array}$$

Higman's Lemma

Graham Higman (1917-2008)

$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$

Crispin Nash-Williams (1932–2001)

bad sequence of words, minimal lengths

$$
\begin{array}{ccccccc}
w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & \ldots \\
|| & || & || & || & || & || & \\
s_0 v_0 & s_1 v_1 & s_2 v_2 & s_3 v_3 & s_4 v_4 & s_5 v_5 & \ldots \\
 & & & & & & \\
 & & s_2 & \leq & s_4 & \leq & \ldots
\end{array}
$$

Higman's Lemma

Graham Higman (1917-2008)

$(S, \leq)$ well-partial-order $\rightsquigarrow$ so is $(S^*, \leq)$
with $s_0 s_1 \cdots s_{k-1} \leq t_0 t_1 \cdots t_{l-1} :\Leftrightarrow$
$\exists$ increasing $\pi : [k] \to [l]$ with $s_i \leq t_{\pi(i)}$

Crispin Nash-Williams (1932–2001)

bad sequence of words, minimal lengths

$$\begin{array}{lcccccccc}
 & w_0 & w_1 & w_2 & w_3 & w_4 & w_5 & \ldots \\
 & \| & \| & \| & \| & \| & \| & \\
 & s_0 v_0 & s_1 v_1 & s_2 v_2 & s_3 v_3 & s_4 v_4 & s_5 v_5 & \ldots \\
 & & & s_2 & \leq & s_4 & \leq & \ldots \\
\rightsquigarrow & s_0 v_0 & s_1 v_1 & v_2 & & v_4 & & \ldots
\end{array}$$

smaller bad sequence! (e.g. $s_0 v_0 \leq v_2 \Rightarrow w_0 \leq w_2$) $\square$

Fundamental Theorem

$$R := K\begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ \vdots & \vdots & \vdots & \\ x_{k-1,0} & x_{k-1,1} & x_{k-1,2} & \cdots \end{bmatrix}$$

polynomial ring

Fundamental Theorem

$$R := K\begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ \vdots & \vdots & \vdots & \\ x_{k-1,0} & x_{k-1,1} & x_{k-1,2} & \cdots \end{bmatrix}$$

polynomial ring

$\mathrm{Inc}(\mathbb{N}) := \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(0) < \pi(1) < \ldots\}$
monoid

Fundamental Theorem

$$R := K\begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ \vdots & \vdots & \vdots & \\ x_{k-1,0} & x_{k-1,1} & x_{k-1,2} & \cdots \end{bmatrix}$$

polynomial ring

$\mathrm{Inc}(\mathbb{N}) := \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(0) < \pi(1) < \ldots\}$
monoid

$\mathrm{Inc}(\mathbb{N})$ acts on R by $\pi x_{ij} = x_{i\pi(j)}$

Fundamental Theorem

$$R := K\begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots \\ \vdots & \vdots & \vdots & \\ x_{k-1,0} & x_{k-1,1} & x_{k-1,2} & \cdots \end{bmatrix}$$

polynomial ring

$\mathrm{Inc}(\mathbb{N}) := \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(0) < \pi(1) < \ldots\}$
monoid

$\mathrm{Inc}(\mathbb{N})$ acts on R by $\pi x_{ij} = x_{i\pi(j)}$

Cohen, Aschenbrenner/Hillar/Sullivant
$f_0, f_1, \ldots \in R \rightsquigarrow \exists i : \forall j \geq i :$
$f_j \in R \cdot \mathrm{Inc}(\mathbb{N}) f_0 + \ldots + R \cdot \mathrm{Inc}(\mathbb{N}) f_{i-1}$

(R *Inc*$(\mathbb{N})$-*Noetherian* $\rightsquigarrow$ Sym$(\mathbb{N})$-Noetherian)

Proof of fundamental theorem

Proof of fundamental theorem

choose well-order $\prec$ on monomials with
$v \prec w \Rightarrow uv \prec uw, \pi(v) \prec \pi(w)$
(e.g. lex with $x_{ij} \prec x_{lm}$ if $j < m$ or $j = m$ and $i < l$)

Proof of fundamental theorem

choose well-order $\prec$ on monomials with
$v \prec w \Rightarrow uv \prec uw, \pi(v) \prec \pi(w)$
(e.g. lex with $x_{ij} \prec x_{lm}$ if $j < m$ or $j = m$ and $i < l$)

bad sequence, minimal leading monomials:

$$\begin{array}{lcccc} & f_0 & f_1 & f_2 & \cdots \\ \mathrm{lm} & \downarrow & \downarrow & \downarrow & \\ & u_0 & u_1 & u_2 & \cdots \end{array}$$

Proof of fundamental theorem

choose well-order $\prec$ on monomials with
$v \prec w \Rightarrow uv \prec uw, \pi(v) \prec \pi(w)$
(e.g. lex with $x_{ij} \prec x_{lm}$ if $j < m$ or $j = m$ and $i < l$)

bad sequence, minimal leading monomials:

$$\begin{tikzcd} f_0 \arrow[d, "\mathrm{lm}"'] & f_1 \arrow[d] & f_2 \arrow[d] & \cdots \\ u_0 & u_1 & u_2 & \cdots \end{tikzcd}$$

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

Proof of fundamental theorem

choose well-order $\prec$ on monomials with
$v \prec w \Rightarrow uv \prec uw, \pi(v) \prec \pi(w)$
(e.g. lex with $x_{ij} \prec x_{lm}$ if $j < m$ or $j = m$ and $i < l$)

bad sequence, minimal leading monomials:

$$\begin{array}{cccc} f_0 & f_1 & f_2 & \cdots \\ \mathrm{lm}\downarrow & \downarrow & \downarrow & \\ u_0 & u_1 & u_2 & \cdots \end{array}$$

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$\Rightarrow$ replace $f_j \rightsquigarrow f_j - t\pi(f_i)$, contradiction

Proof of claim

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$u_0 \qquad u_1 \qquad u_2 \quad \cdots$ monomials in x_{ij}, $i \in [k]$, $j \in \mathbb{N}$

Proof of claim

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$$\begin{array}{cccccl} u_0 & & u_1 & & u_2 & \cdots \quad \text{monomials in } x_{ij}, i \in [k], j \in \mathbb{N} \\ \downarrow & & \downarrow & & \downarrow & \\ \alpha_0 & & \alpha_1 & & \alpha_2 & \cdots \quad \text{finite words over the alphabet } \mathbb{N}^k \end{array}$$

Example with $k = 2$

$$u_i = x_{00} x_{10}^2 x_{01}^3 \rightsquigarrow \alpha_i = \begin{matrix} 1 & 3 \\ 2 & 0 \end{matrix}$$

Proof of claim

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$$\begin{array}{cccccc} u_0 & & u_1 & & u_2 & \cdots \quad \text{monomials in } x_{ij},\ i \in [k],\ j \in \mathbb{N} \\ \downarrow & & \downarrow & & \downarrow & \\ \alpha_0 & & \alpha_1 & & \alpha_2 & \cdots \quad \text{finite words over the alphabet } \mathbb{N}^k \end{array}$$

Example with $k = 2$

$$u_i = x_{00} x_{10}^2 x_{01}^3 \rightsquigarrow \alpha_i = \begin{array}{cc} 1 & 3 \\ 2 & 0 \end{array}$$

Higman's lemma with $S = \mathbb{N}^k$ (Dixon) $\rightsquigarrow \exists i < j : \alpha_i \leq \alpha_j$

Proof of claim

Claim
$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$$\begin{tikzcd} u_0 \arrow[d] & u_1 \arrow[d] & u_2 \arrow[d] & \cdots & \text{monomials in } x_{ij},\ i \in [k],\ j \in \mathbb{N} \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \text{finite words over the alphabet } \mathbb{N}^k \end{tikzcd}$$

Example with $k = 2$

$$u_i = x_{00} x_{10}^2 x_{01}^3 \rightsquigarrow \alpha_i = \begin{matrix} 1 & 3 \\ 2 & 0 \end{matrix} \leq \begin{matrix} 1 & 0 & 3 \\ 3 & 1 & 0 \end{matrix} = \alpha_j$$

Higman's lemma with $S = \mathbb{N}^k$ (Dixon) $\rightsquigarrow \exists i < j : \alpha_i \leq \alpha_j$

Proof of claim

Claim

$\exists i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : \pi(u_i) | u_j$

$$\begin{tikzcd}
u_0 \arrow[d] & u_1 \arrow[d] & u_2 \arrow[d] & \cdots & \text{monomials in } x_{ij}, i \in [k], j \in \mathbb{N} \\
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \text{finite words over the alphabet } \mathbb{N}^k
\end{tikzcd}$$

Example with $k = 2$

$$u_i = x_{00} x_{10}^2 x_{01}^3 \rightsquigarrow \alpha_i = \begin{matrix} 1 & 3 \\ 2 & 0 \end{matrix} \leq \begin{matrix} 1 & 0 & 3 \\ 3 & 1 & 0 \end{matrix} = \alpha_j$$

Higman's lemma with $S = \mathbb{N}^k$ (Dixon) $\rightsquigarrow \exists i < j : \alpha_i \leq \alpha_j$

$\rightsquigarrow \exists \pi \in \mathrm{Inc}(\mathbb{N}) : \pi u_i | u_j$ $\square$

Equivariantly Noetherian

Definition
Π monoid acting on ring $R \rightsquigarrow R$ Π-*Noetherian* if each chain $I_0 \subseteq I_1 \subseteq \ldots$ of Π-stable ideals stabilises

Equivariantly Noetherian

Definition
Π monoid acting on ring $R \rightsquigarrow R$ Π-*Noetherian* if each chain $I_0 \subseteq I_1 \subseteq \ldots$ of Π-stable ideals stabilises

Lemma
if R is Π-Noetherian and

- $I \subseteq R$ is Π-stable $\rightsquigarrow R/I$ is Π-Noetherian

Equivariantly Noetherian

Definition
Π monoid acting on ring $R \rightsquigarrow R$ Π-*Noetherian* if each chain $I_0 \subseteq I_1 \subseteq \dots$ of Π-stable ideals stabilises

Lemma
if R is Π-Noetherian and

- $I \subseteq R$ is Π-stable $\rightsquigarrow R/I$ is Π-Noetherian
- $R = S \oplus M$, S subring and M S-module, both Π-stable $\rightsquigarrow S$ is Π-Noetherian

Equivariantly Noetherian

Definition
Π monoid acting on ring $R \rightsquigarrow R$ Π-*Noetherian* if each chain $I_0 \subseteq I_1 \subseteq \ldots$ of Π-stable ideals stabilises

Lemma
if R is Π-Noetherian and

- $I \subseteq R$ is Π-stable $\rightsquigarrow R/I$ is Π-Noetherian
- $R = S \oplus M$, S subring and M S-module, both Π-stable $\rightsquigarrow S$ is Π-Noetherian

$I \subseteq S$ ideal, $J := IR \subseteq R$ ideal generated by I
$\rightsquigarrow J \cap S = I$ (apply projection $R \to S$ to $s = \sum_\ell i_\ell r_\ell$)

Equivariantly Noetherian

Definition
Π monoid acting on ring $R \rightsquigarrow R$ Π-*Noetherian* if each chain $I_0 \subseteq I_1 \subseteq \ldots$ of Π-stable ideals stabilises

Lemma
if R is Π-Noetherian and

- $I \subseteq R$ is Π-stable $\rightsquigarrow R/I$ is Π-Noetherian
- $R = S \oplus M$, S subring and M S-module, both Π-stable $\rightsquigarrow S$ is Π-Noetherian

$I \subseteq S$ ideal, $J := IR \subseteq R$ ideal generated by I
$\rightsquigarrow J \cap S = I$ (apply projection $R \to S$ to $s = \sum_\ell i_\ell r_\ell$)
$\rightsquigarrow$ if $J_n = J_{n+1} = \ldots$ then $I_n = J_n \cap S = J_{n+1} \cap S = I_{n+1} = \ldots$

$\square$

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i, j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i, j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

Theorem
K characteristic 0, $k \in \mathbb{N} \rightsquigarrow K[y_{ij}]/\langle (k+1) \times (k+1)\text{-minors}\rangle$
is $\mathrm{Inc}(\mathbb{N})$-Noetherian

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i,j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

Theorem
K characteristic 0, $k \in \mathbb{N} \rightsquigarrow K[y_{ij}]/\langle (k+1) \times (k+1)\text{-minors}\rangle$
is $\mathrm{Inc}(\mathbb{N})$-Noetherian

$$K[y_{ij}] \xrightarrow{\quad y_{ij} \mapsto \sum_\ell x_{i\ell} z_{\ell j} \quad} K[x_{i\ell}, z_{\ell j} \mid \ell \in [k], i, j \in \mathbb{N}] =: R$$

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i, j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

Theorem
K characteristic 0, $k \in \mathbb{N} \rightsquigarrow K[y_{ij}]/\langle (k+1) \times (k+1)\text{-minors} \rangle$
is $\mathrm{Inc}(\mathbb{N})$-Noetherian

$$\begin{tikzcd}
K[y_{ij}] \arrow[r, "y_{ij} \mapsto \sum_\ell x_{i\ell} z_{\ell j}"] \arrow[d] & K[x_{i\ell}, z_{\ell j} \mid \ell \in [k], i, j \in \mathbb{N}] =: R \\
K[y_{ij}]/\langle \text{minors} \rangle \arrow[ur, hook] &
\end{tikzcd}$$

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i, j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

Theorem
K characteristic 0, $k \in \mathbb{N} \rightsquigarrow K[y_{ij}]/\langle (k+1) \times (k+1)\text{-minors}\rangle$
is $\mathrm{Inc}(\mathbb{N})$-Noetherian

$$\begin{tikzcd}
K[y_{ij}] \arrow[r, "y_{ij} \mapsto \sum_\ell x_{i\ell} z_{\ell j}"] \arrow[d] & K[x_{i\ell}, z_{\ell j} \mid \ell \in [k], i, j \in \mathbb{N}] =: R \\
K[y_{ij}]/\langle \text{minors} \rangle \arrow[r, "\cong\ (\text{FFT} + \text{SFT for } \mathrm{GL}_k)"'] \arrow[ur, hook] & R^{\mathrm{GL}_k} \text{ (invariants)} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

I: Bounded-rank matrices

$\mathrm{Inc}(\mathbb{N})$ acts on $K[y_{ij} \mid i, j \in \mathbb{N}]$ by $\pi y_{ij} = y_{\pi(i)\pi(j)}$

not $\mathrm{Inc}(\mathbb{N})$-Noetherian: $y_{12}y_{21}, y_{12}y_{23}y_{31}, \ldots$

Theorem
K characteristic 0, $k \in \mathbb{N} \rightsquigarrow K[y_{ij}]/\langle (k+1) \times (k+1)\text{-minors}\rangle$
is $\mathrm{Inc}(\mathbb{N})$-Noetherian

$$\begin{tikzcd}
K[y_{ij}] \arrow[r, "y_{ij} \mapsto \sum_\ell x_{i\ell} z_{\ell j}"] \arrow[d] & K[x_{i\ell}, z_{\ell j} \mid \ell \in [k], i, j \in \mathbb{N}] =: R \\
K[y_{ij}]/\langle \text{minors} \rangle \arrow[r, "\cong\ (\text{FFT} + \text{SFT for } \mathrm{GL}_k)"'] \arrow[ur, bend left] & R^{\mathrm{GL}_k} \text{ (invariants)} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

M direct sum of non-trivial reps $\rightsquigarrow R = R^{\mathrm{GL}_k} \oplus M$
R $\mathrm{Inc}(\mathbb{N})$-Noetherian $\rightsquigarrow$ so is R^H $\square$

Sampling contingency tables

Fixed row and column sums

$A, B \in \mathbb{N}^{m \times n}$ with $a_{i+} = b_{i+}$ and $a_{+j} = b_{+j}$

$\Rightarrow \exists A = A_0, A_1, \ldots, A_k = B \in \mathbb{N}^{m \times n}$ with

$$A_l - A_{l-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\rightsquigarrow$ moves "independent" of m, n.

Sampling contingency tables

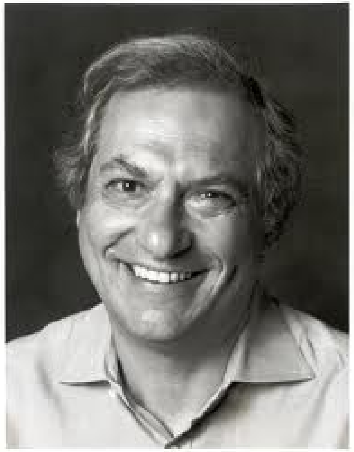
Fixed row and column sums

$A, B \in \mathbb{N}^{m \times n}$ with $a_{i+} = b_{i+}$ and $a_{+j} = b_{+j}$

$\Rightarrow \exists A = A_0, A_1, \ldots, A_k = B \in \mathbb{N}^{m \times n}$ with

$$A_l - A_{l-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\rightsquigarrow$ moves "independent" of m, n.



Diaconis-Sturmfels

Markov moves = generating set of toric ideal

(ker[$y_{ij} \mapsto x_i z_j$] generated by $\{y_{ij} y_{i'j'} - y_{ij'} y_{i'j}\}$)

II: Independent set theorem

Hillar/Sullivant/Hosten

F family of subsets of $[m]$

$y(i_1, \ldots, i_m)$ and $x(S, (i_s)_{s \in S})$ for $S \in F$ variables

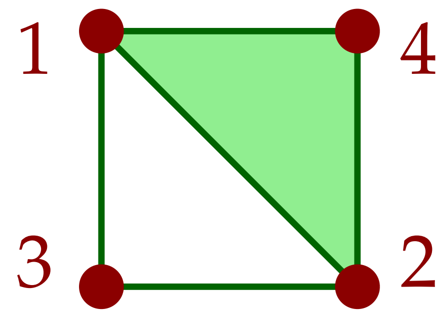
$I := \ker[y(i_1, \ldots, i_m) \mapsto \prod_{A \in S} x(S, (i_s)_{s \in S})]$

Example

$m = 4, F = \{124, 13, 23\}$

variables $y(abcd), x(abd), z(ac), u(bc)$

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$



II: Independent set theorem

Hillar/Sullivant/Hosten

F family of subsets of $[m]$

$y(i_1, \ldots, i_m)$ and $x(S, (i_s)_{s \in S})$ for $S \in F$ variables

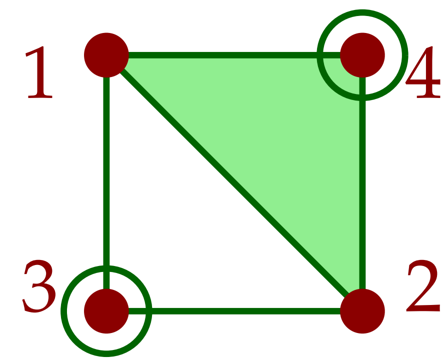
$I := \ker[y(i_1, \ldots, i_m) \mapsto \prod_{A \in S} x(S, (i_s)_{s \in S})]$

Example

$m = 4, F = \{124, 13, 23\}$

variables $y(abcd), x(abd), z(ac), u(bc)$

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$



Theorem

$T \subseteq [m]$ *independent set* ($|T \cap S| \leq 1$ for $S \in F$)

$i_t, t \in T$ run through $\mathbb{N}$ and $i_t, t \notin T$ through $[r_t]$

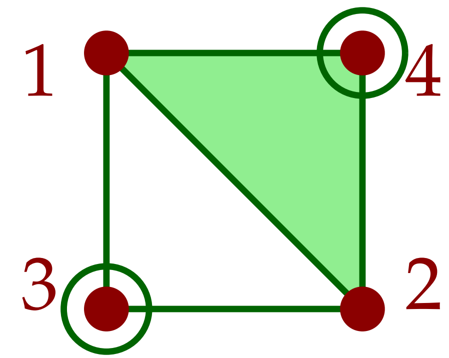
$\rightsquigarrow I$ generated by finitely many $\mathrm{Inc}(\mathbb{N})$-orbits

Proof of independent set theorem

Example

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$

$i_1 \in [r_1], i_2 \in [r_2], i_3, i_4 \in \mathbb{N}$

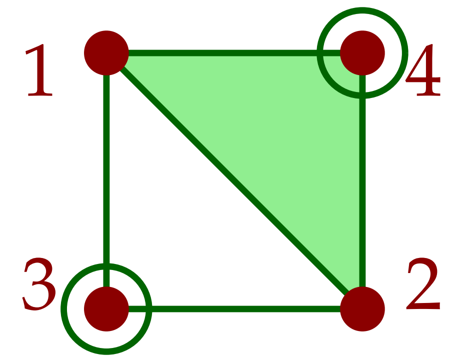


Proof of independent set theorem

Example

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$

$i_1 \in [r_1], i_2 \in [r_2], i_3, i_4 \in \mathbb{N}$



$$K[y(abc)] \longrightarrow R = K[v(abc), w(abd)] \longrightarrow K[x(abd), z(ac), u(bc)]$$

$y(abcd) \mapsto v(abc)w(abd)$

$v(abc) \mapsto z(ac)u(bc)$

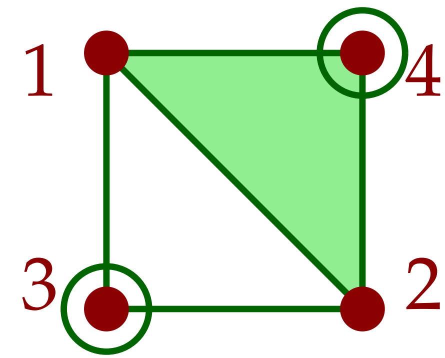
$w(abd) \mapsto x(abd)$

Proof of independent set theorem

Example

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$

$i_1 \in [r_1], i_2 \in [r_2], i_3, i_4 \in \mathbb{N}$



$$\begin{tikzcd}
K[y(abc)] \arrow[r] \arrow[d] & R = K[v(abc), w(abd)] \arrow[r] & K[x(abd), z(ac), u(bc)] \\
K[y(abc)]/J & &
\end{tikzcd}$$

$y(abcd) \mapsto v(abc)w(abd)$

$v(abc) \mapsto z(ac)u(bc)$
$w(abd) \mapsto x(abd)$

$J = \langle y(abcd)y(abc'd') - y(abcd')y(abc'd)\rangle \subseteq I$

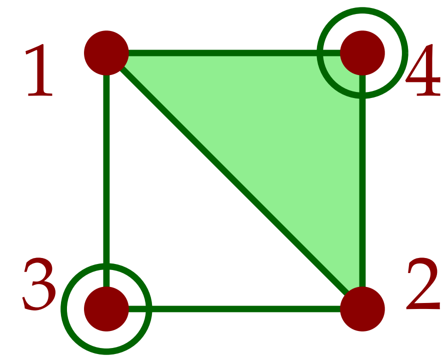
$\rightsquigarrow r_1 \cdot r_2$ copies of Segre product

Proof of independent set theorem

Example

$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$

$i_1 \in [r_1], i_2 \in [r_2], i_3, i_4 \in \mathbb{N}$



$$\begin{tikzcd}
K[y(abc)] \arrow[r] \arrow[d, "y(abcd) \mapsto v(abc)w(abd)"] & R = K[v(abc), w(abd)] \arrow[r, "{v(abc) \mapsto z(ac)u(bc),\ w(abd) \mapsto x(abd)}"] & K[x(abd), z(ac), u(bc)] \\
K[y(abc)]/J \arrow[r, "\cong"] & R^{K^*} \arrow[u, hook] &
\end{tikzcd}$$

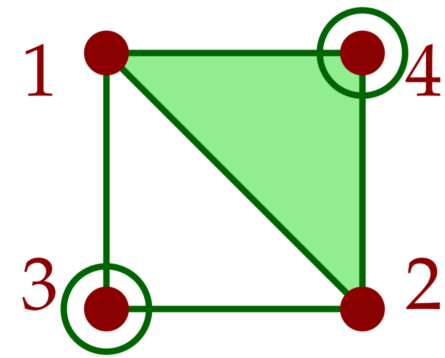
$J = \langle y(abcd)y(abc'd') - y(abcd')y(abc'd) \rangle \subseteq I$

$\rightsquigarrow r_1 \cdot r_2$ copies of Segre product

K^* acts on R by $t \cdot v(abc) = tv(abc),\ t \cdot w(abd) = t^{-1}w(abd)$

Proof of independent set theorem

Example



$I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]$

$i_1 \in [r_1], i_2 \in [r_2], i_3, i_4 \in \mathbb{N}$

$$\begin{tikzcd}
K[y(abc)] \arrow[r] \arrow[d, two heads, "y(abcd) \mapsto v(abc)w(abd)"] & R = K[v(abc), w(abd)] \arrow[r, "\substack{v(abc) \mapsto z(ac)u(bc) \\ w(abd) \mapsto x(abd)}"] & K[x(abd), z(ac), u(bc)] \\
K[y(abc)]/J \arrow[r, "\cong"] & R^{K^*} \arrow[u, hook] &
\end{tikzcd}$$

$J = \langle y(abcd)y(abc'd') - y(abcd')y(abc'd) \rangle \subseteq I$

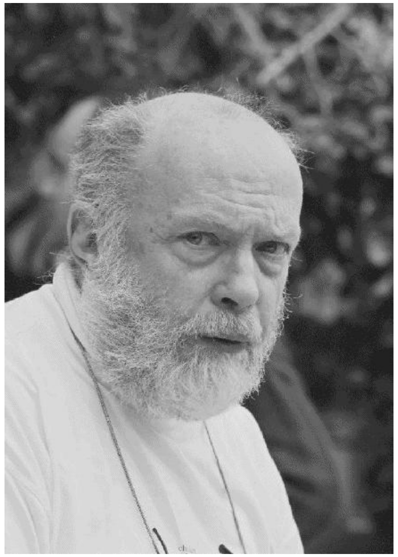
$\rightsquigarrow r_1 \cdot r_2$ copies of Segre product

K^* acts on R by $t \cdot v(abc) = tv(abc),\ t \cdot w(abd) = t^{-1}w(abd)$

$R = R^{K^*} \oplus M \rightsquigarrow R^{K^*}$ Inc($\mathbb{N}$)-Noeth $\square$

III: Vandermonde relations

Andreas Dress



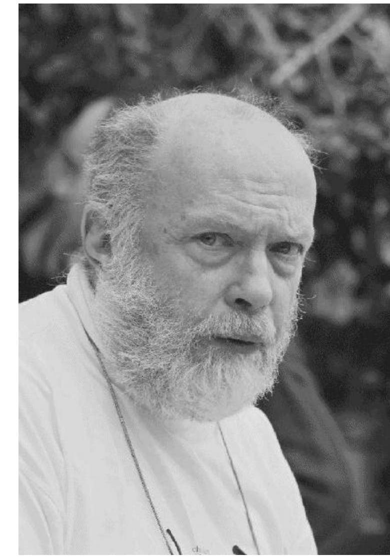
$y_0, y_1, \ldots$ variables

$z_I := \prod_{i,j \in I, i<j} (y_i - y_j)$ for $|I| = k$, fixed

Relations among the z_I finite up to $\mathrm{Sym}(\mathbb{N})$*?*

III: Vandermonde relations

Andreas Dress



$y_0, y_1, \ldots$ variables
$z_I := \prod_{i,j \in I, i<j} (y_i - y_j)$ for $|I| = k$, fixed

Relations among the z_I finite up to $\mathrm{Sym}(\mathbb{N})$*?*

Theorem
Yes (in characteristic zero).

$$z_I = \det \begin{bmatrix} 1 & \cdots & 1 \\ y_{i_0} & \cdots & y_{i_{k-1}} \\ \vdots & & \vdots \\ y_{i_0}^{k-1} & \cdots & y_{i_{k-1}}^{k-1} \end{bmatrix}$$ satisfy *Plücker relations*.

mod these $\rightsquigarrow$
invariant ring $K[x_{ij}, i \in [n], j \in \mathbb{N}]^{\mathrm{SL}_n}$ is $\mathrm{Inc}(\mathbb{N})$-Noetherian $\square$