Fewestnomials in determinantal ideals

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joint with Thomas Kahle and Finn Wiersig

Fewestnomials in ideals

Definition

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$\rightsquigarrow c(I) := \min\{\text{number of terms in } f \mid f \in I \setminus \{0\}\}$

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Questions

- What is $c(I)$ for your favourite I?
- Is $c(I)$ computable from generators of I?
- Why should you care?

The curious case of $c(I) = 2$

Philosophy [Eisenbud-Sturmfels, 1996]

Ideals *spanned* by binomials have many beautiful properties.

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For $n \in \mathbb{Z}_{\geq 1}$, $I_n := ((x-z)^2, nx - y - (n-1)z) \subseteq \mathbb{Q}[x, y, z]$ has $c(I_n) = 2$ and $x^n - yz^{n-1}$ is the lowest-degree binomial in I_n.

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$\exists$ algorithm that, on input $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$, decides $c(I) \stackrel{?}{=} 2$.

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Question: Is $c(I)$ computable?
Special cases: What if $n = 1$ or if I is linearly generated?

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Outline of the Jensen-Kahle-Kathän algorithm:

- Rule out $c(I) = 1$, pass to $I \subseteq \mathbb{Q}[x_1^{\pm}, \ldots, x_n^{\pm}]$.
- If $x^\alpha - a \cdot x^0 \in I$, then $\mathrm{Trop}(I) \subseteq \alpha^\perp$.
- Compute a basis $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}^n$ of $\mathrm{Trop}(I)^\perp$.
- After a monomial coordinate change: $\alpha_i = e_i$, so $x^{\alpha_i} = x_i$.
- Need to look for binomials in $J := \mathbb{Q}[x_1^{\pm}, \ldots, x_m^{\pm}] \cap I$.

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- $\mathrm{Trop}(J) = \{0\} \subseteq \mathbb{R}^m$.
- Hence $A := \mathbb{Q}[x_1^{\pm}, \ldots, x_m^{\pm}]/J$ is finite-dimensional.
- Let $M_i \in \mathrm{End}(A)$ be multiplication with x_i.
- Then need to find $\alpha \in \mathbb{Z}^m$ s.t. $M_1^{\alpha_1} \cdots M_m^{\alpha_m} = a \cdot \mathrm{id}_A$.
- This had already been solved in number theory.

Our results

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For r even, $X := \{A \mid A^T = -A, \mathrm{rk}(A) \leq r\} \subseteq K^{m(m-1)/2}$ we have $c(X) = (r+1)!! = (r+1)(r-1)(r-3)\cdots 3 \cdot 1$.

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... and in each case we know all $f \in I(X)$ with $c(X)$ terms.

Theorem 1 $\Rightarrow$ Theorem 2

Know: for a very general r-space $X \subseteq K^n$, $c(X) = r + 1$.
To show: $c(\{rk(A) \leq r\}) \geq (r + 1)!$

- Induction: assume true for $(m - 1, r)$ and $(m - 1, r - 1)$.

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- Take $A \in K^{(m-1) \times n}$ very general of rank r. Then $f_\alpha(A) \neq 0$ for all α with $f_\alpha \neq 0$. Theorem 1 applied to the row space X of A yields that $f_\alpha \neq 0$ for at least $(r + 1)$ distinct α.

$\rightsquigarrow$ f has at least $(r + 1) \cdot r! = (r + 1)!$ terms $\square$

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Then X defines the uniform matroid of rank 3 on K^6, but the 4×4-Pfaffian $x_{12}x_{24} - x_{13}x_{24} + x_{14}x_{23}$ vanishes on X and has $3 < 4$ terms.

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• Hence F' contains a monomial x^β divisible by some x_j, $j > r$, but then $(|F| =)|F'| \geq r + 1$. $\square$

Characterisation of equality

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For $r \geq 2$, the only $(r+1)$-term polynomials that vanish on a very general r-space $X \subseteq K^n$ are $c \cdot x^\alpha \cdot \ell^{p^e}$ where ℓ is a linear form with $r+1$ terms.

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Theorem 3'. $(r+2)$-Pfaffians in the skew-symmetric case.

Key proof ingredient

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reverse lexicographic order $\succ$ defined by $x^{\alpha} \succ x^{\beta}$ if the *largest i* with $\alpha_i \neq \beta_i$ satisfies $\alpha_i < \beta_i$.

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If $s \geq 3$, char $K = 0$ and a subspace $F \subseteq K[x_1, ..., x_n]_d$ has $\text{gin}_\succ F = x_1^{d-1} \cdot \langle x_1, ..., x_s \rangle$, then $F = f \cdot \langle \ell_1, ..., \ell_s \rangle$ for some $f \in K[x_1, ..., x_n]_{d-1}$ and some linear forms $\ell_1, ..., \ell_s$.

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Open problem: the case of symmetric matrices!

Hitting set generators (after Robert Andrews)

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For a family of polynomials $S \subseteq K[x_1, \ldots, x_n]$, a *hitting set generator* is a polynomial map $g : K^m \to K^n$ such that $f \circ g \in K[y_1, \ldots, y_m] \setminus \{0\}$ for all $f \in S \setminus \{0\}$.

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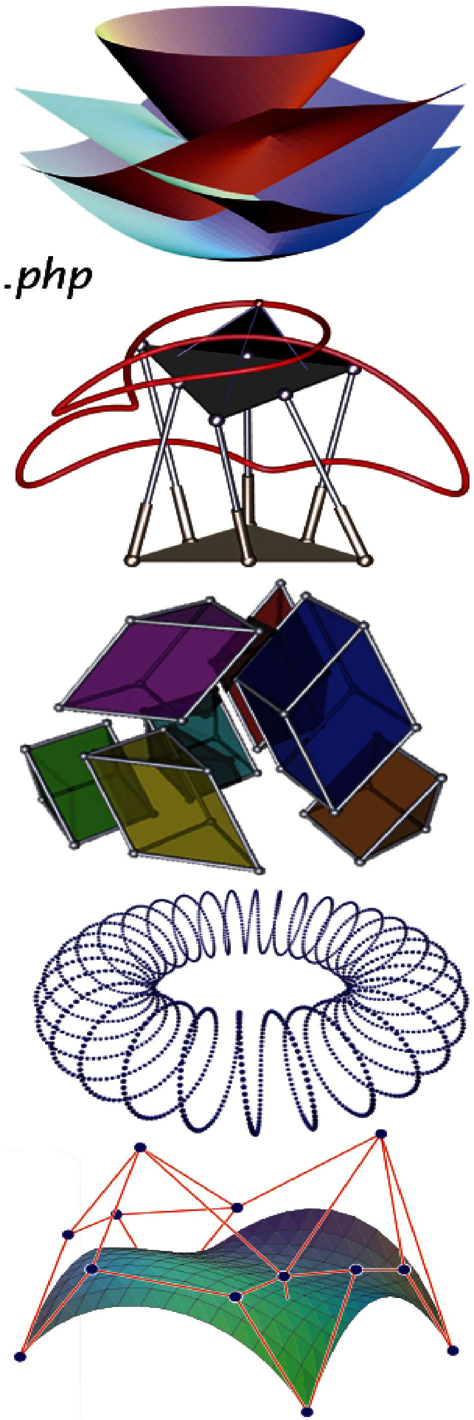
Observation [Robert Andrews]
For $S = \{\text{polynomials with} \leq t \text{ terms}\}$, choose r such that $(r+1)! \geq t$, Theorem 2 gives a degree-two hitting set generator $g : K^{\sqrt{n}\times r} \times K^{r\times\sqrt{n}} \to K^{\sqrt{n}\times\sqrt{n}} = K^n, (A, B) \mapsto AB$.

The resulting $m = c \cdot \sqrt{n} \cdot \log(t) / \log(\log(t))$ is near optimal.



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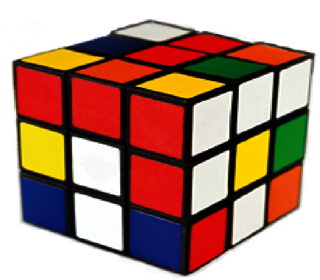

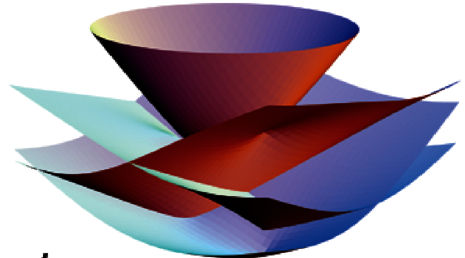

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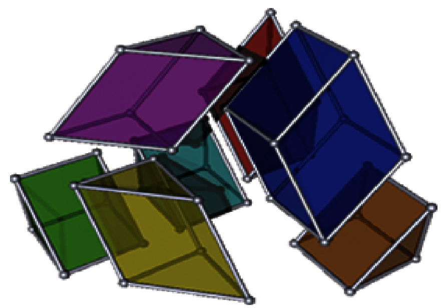

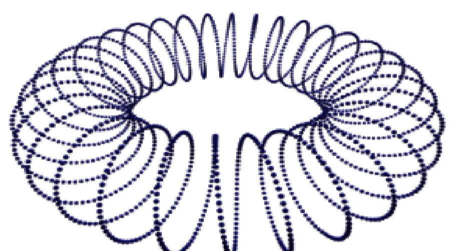

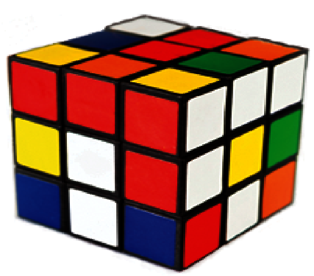

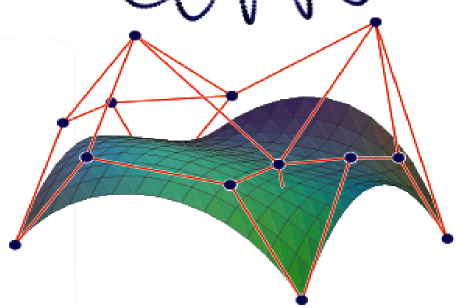

Thank you!