# Constructing Lie algebras from extremal elements

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# Sandwich and extremal elements

$L$ Lie algebra over $K$, char $K \neq 2$
$x \in L$ is **sandwich** if $xxL := [x, [x, L]]) = 0$
$x \in L$ is **extremal** if $xxL \subseteq Kx$
$\Leftrightarrow \exists f_x : L \to K$ linear $: xxy = f_x(y)x$

Goal: analyse "moduli spaces" of Lie algebras with distinguished extremal generators.

# Sandwich algebras

$\Gamma = (\Pi, E)$ finite, undirected graph
$K$ ground field of characteristic $\neq 2$
$F =$ (free Lie algebra over $K$ generated by $\Pi)/(xy$ for $x, y \in \Pi, x \not\sim y)$
$I_0 \subseteq F$ generated by $xxy,\ x \in \Pi, y \in F$
$S := F/I_0$

**Theorem (Kostrikin-Zelmanov)** $S$ is finite-dimensional.

$V$ graded, finite-dimensional with $F = V \oplus I_0$
Note: $V \supseteq K\Pi$

**Example**
$\Gamma = K_3$
$V = \langle x, y, z, xy, xz, yz, xyz, yzx \rangle_K$

# The moduli space

$f = (f_x)_{x \in \Pi} \in (F^*)^\Pi$
$\rightsquigarrow I(f) \subseteq F$ generated by $xxy - f_x(y)x,\ x \in \Pi,\ y \in F$
$L(f) := F/I(f)$
$\rightsquigarrow I(0) = I_0$ and $S = L(0)$

**Proposition (Cohen-Steinbach-Ushirobira-Wales)**
$V + I(f) = F$ for all $f$.

**Definition**
$\mathcal{M}(\Gamma) := \{f \in (F^*)^\Pi \mid \dim L(f) = \dim S\}$
$= \{f \in (F^*)^\Pi \mid F = V \oplus I(f)\}$

moduli space of maximal-dimensional Lie algebras with
extremal generators $\leftrightarrow \Pi$
commutation relations $\leftrightarrow E^c$

# Finiteness

**Theorem (D, in ’t panhuis, Postma, Roozemond)**
**$\mathcal{M}(\Gamma)$ is naturally an affine $K$-variety of finite type.**

Idea:

1. $\mathcal{M}(\Gamma) \to (V^*)^\Pi$,
   $(f_x)_{x\in\Pi} \mapsto f|_V := (f_x|_V)_{x\in\Pi}$ injective
   (can express each $f_x(y),\ x \in X,\ y \in F$ polynomially in $f|_V$)
2. image given by polynomial equations

What does $\mathcal{M}(\Gamma)$ look like?
What Lie algebras does it parameterise?

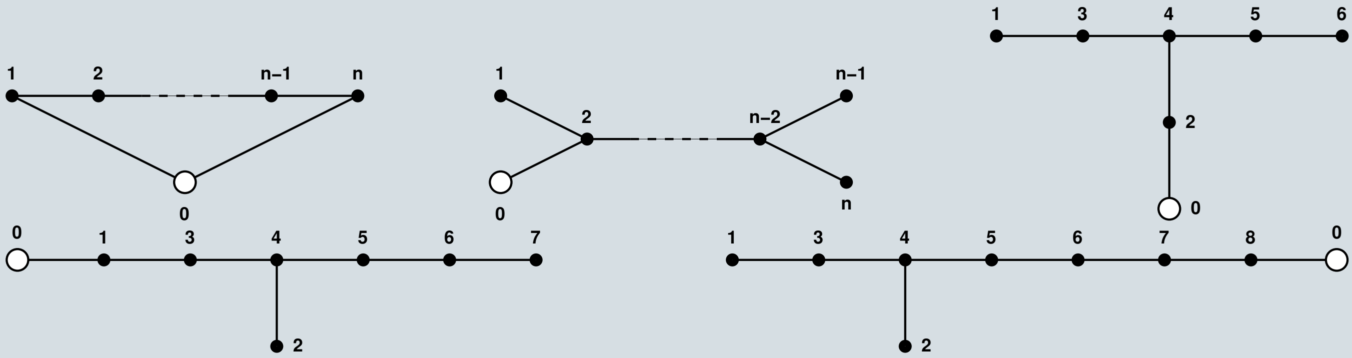
# Example (Cohen-Steinbach-Ushirobira-Wales)

$\Gamma = K_3$
$\mathcal{M}(\Gamma) = \mathbb{A}^4_K$ with coordinates $f_x(y), f_y(z), f_z(x), f_x(yz)$

Possible Lie algebras:

1. $\mathfrak{sl}_3$
2. $\mathfrak{sl}_2 \ltimes K^2 \oplus K^2 \oplus K^1$
3. $\mathfrak{sl}_2 \ltimes K^2 \oplus K^2 \oplus K^1$
4. $\mathfrak{n}_+ \ltimes \mathfrak{g}/\mathfrak{n}_+$

# $\Gamma =$ simply laced affine-type Dynkin diagram




**Theorem (D-in 't panhuis)**
$\mathcal{M}(\Gamma) \cong \mathbb{A}_K^{|E|+1}$ and
$f \in$ open dense subset $\Rightarrow L(f) \cong \mathfrak{g}$ Chevalley of type $\Gamma$

# Proof idea

$\Gamma$ simply laced Dynkin diagram

1. determine $S$ using Kac-Moody$(\Gamma)$
   $\rightsquigarrow S \cong \mathfrak{n}_+ \ltimes \mathfrak{g}/\mathfrak{n}_+$
   $\rightsquigarrow V$ $\mathbb{N}^\Pi$-graded
2. express $f|_V$ polynomially in $f_x(y)(= f_y(x)),\ x \sim y$
   and $f_{x_0}(m)$ for $m \in V$ of highest weight in $\mu \in \mathbb{N}^\Pi$ with $\mu_{x_0} = 0$
   $\rightsquigarrow \mathcal{M}(\Gamma) \subseteq \mathbb{A}^{|E|+1}$ closed
3. find a sufficiently general point $f \in M(\Gamma)$
   with $L(f)$ Chevalley of type $\Gamma$
4. move the point around in an open dense subset of $\mathbb{A}^{|E|+1}$

**Remark:** over $\mathbb{C}$, simple Lie algebras are *rigid*.

# The scaling torus

$T = (K^*)^\Pi$ acts on $M(\Gamma)$
characters on $\mathbb{A}^{|E|+1}$ for $\Gamma$ affine Dynkin:
$\alpha_e := \alpha_x + \alpha_y,\ e = \{x, y\} \in E$
$\delta := \alpha_{x_0} +$ highest root
(actually, $-\alpha_e, -\delta$)

1. $D^{(1)}_{\text{even}}, E^{(1)}_7, E^{(1)}_8$: independent characters
2. $A^{(1)}_{\text{even}}, D^{(1)}_{\text{odd}}, E^{(1)}_6$: $\alpha_e,\ e \in E$ independent, $\delta$ in their $\mathbb{Q}$-span
3. $A^{(1)}_{\text{odd}}$: $\alpha_e,\ e \in E$ dependent and $\delta$ in their $\mathbb{Q}$-span.

# What about

1. other graphs? (in ’t panhuis-Postma-Roozemond)
2. equations for the complement of the open dense set?
3. recognition?
4. structure of $\mathcal{M}(\Gamma)$ in general?