Where innovation starts

Tropical reparameterisations

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Two ways to describe a line

implicitly, by equations

$$X := \{(x, y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2$$

explicitly, by parameterisation

$$\phi : \mathbb{A}^1 \to \mathbb{A}^2, \quad u \mapsto (u, u + 1); \quad X = \operatorname{im} \phi$$

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Tropicalising those two ways

by equations

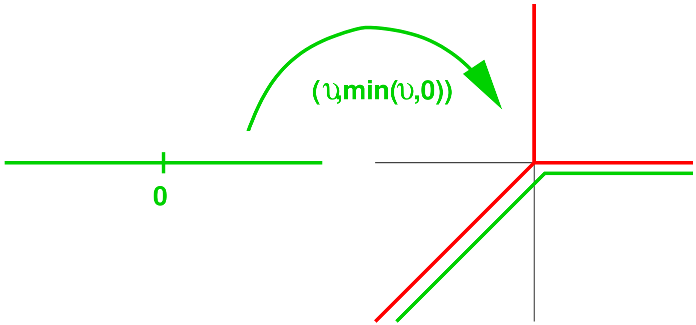
$X = \{(x,y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2$

$\mathcal{T}X = \{(\xi,\eta) \mid \min\{\eta,\xi,0\}$ **attained** $\geq$ **twice**$\}$
$\subset \mathbb{R}^2_\infty$

by parameterisation

$\phi : u \mapsto (u, u+1)$

$\mathcal{T}\phi : v \mapsto (v, \min\{v,0\})$



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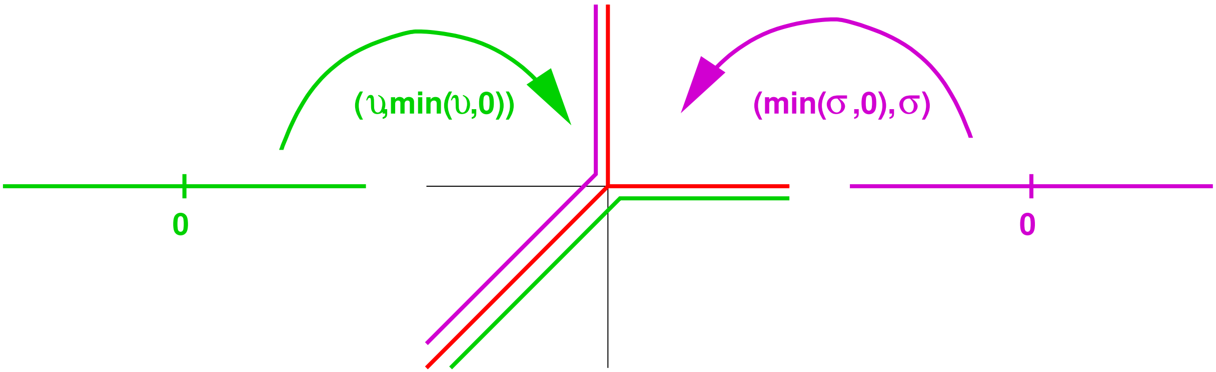
Reparameterisation for the line

tropicalisation and composition don’t commute

$$\alpha : \mathbb{A}^1 \to \mathbb{A}^1, \quad s \mapsto s - 1$$
$$\phi' := \phi \circ \alpha : \mathbb{A}^1 \to \mathbb{A}^2, \quad s \mapsto (s-1, s)$$
$$\mathcal{T}(\phi') : \mathbb{R}_\infty \to \mathbb{R}^2_\infty, \quad \sigma \mapsto (\min\{\sigma, 0\}, \sigma)$$



Tropicalising..

polynomials:

$$f = \sum_\alpha c_\alpha x^\alpha \in K[x] \rightsquigarrow \mathcal{T} f = \min_\alpha v(c_\alpha) + \langle \xi, \alpha \rangle$$

varieties:

$X \subseteq \mathbb{A}^n$ variety

$I = I(X) \subseteq K[x]$

$\mathcal{T} X := \{\xi \in \mathbb{R}^n_\infty \mid \forall f \in I : \mathcal{T} f$ not linear at $\xi\}$

tropicalisation of X

maps?

easy special case:

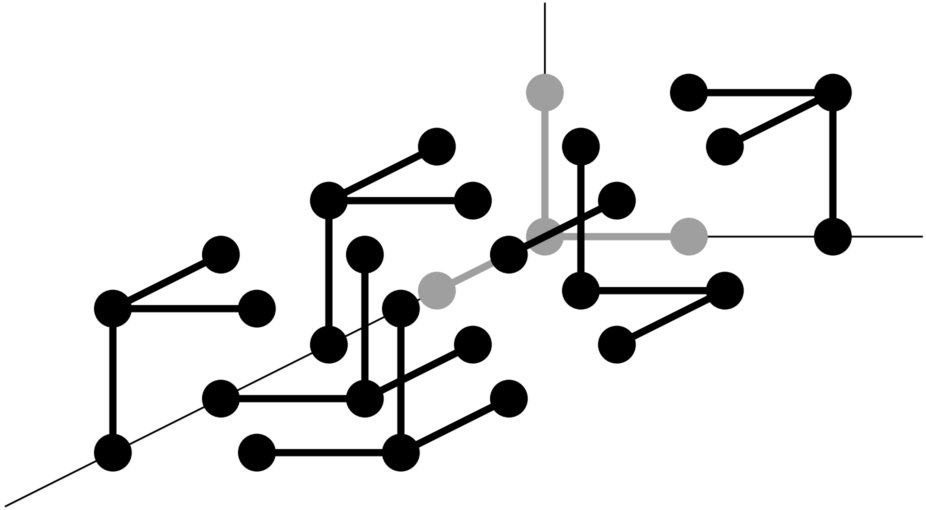
$\phi : \mathbb{A}^m \to X \subseteq \mathbb{A}^n$

$\rightsquigarrow \operatorname{im}(\mathcal{T}\phi : \mathbb{R}^m_\infty \to \mathbb{R}^n_\infty) \subseteq \mathcal{T} X$

Useful?

Theorem (Baur and D)

computed all secant dimensions of $\mathbb{P}^1 \times \mathbb{P}^2$ and $\mathcal{F} := \{\text{point} \subset \text{line} \subset \mathbb{P}^2\}$ in all $\mathrm{SL}_2 \times \mathrm{SL}_3$ resp. SL_3 equivariant embeddings.



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Reparameterisations

four questions

$\phi : \mathbb{A}^m \to \overline{\mathrm{im}\,\phi} = X \subseteq \mathbb{A}^n$ polynomial map
$\exists?$ finitely many *(or one)* $\alpha_i : \mathbb{A}^{p_i} \to \mathbb{A}^m$
(or rational maps) such that $\cup_i \mathrm{im}\,\mathcal{T}(\phi \circ \alpha_i) = \mathcal{T}(X)$

remark

Sturmfels-Tevelev-Yu (2007) describe $\mathcal{T}X$ from ϕ in case of generic coefficients; generalisations use Hacking-Keel-Tevelev's *geometric tropicalisation* (2007).

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Two reductions

lemma

If $\phi = (\phi_1, \ldots, \phi_n)$ with all ϕ_i homogeneous of same degree, then the four questions are equivalent.
(Multiply with common denominator;
combine several reparameterisations into one.)

observation

All four questions reduce to the case where $\operatorname{codim} X \in \{0, 1\}$.
(Otherwise choose generic mononomial map
$\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ where $d = \dim X$.)

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Toric varieties and linear spaces

toric varieties

$\phi : \mathbb{A}^m \to X \subseteq \mathbb{A}^n$ *monomial*
$\rightsquigarrow \mathcal{T}\phi$ linear and $\operatorname{im} \mathcal{T}\phi = \mathcal{T}X$,
a linear space in $\mathbb{R}^n_\infty$.

linear spaces (Yu-Yuster, 2006)

$\phi : \mathbb{A}^m \to X \subseteq \mathbb{A}^n$ *linear*, given by matrix ϕ
$\rightsquigarrow \operatorname{im} \mathcal{T}\phi = \mathcal{T}X$ iff every $v \in X$ of minimal support (*cocircuit*) is scalar multiple of a column of ϕ.

(Can be achieved by precomposing ϕ with a linear map.)

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Example

$$\phi = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix}$$ **over** $\mathbb{C}((t))$**;** $X = \{x + t^2 y - tz = 0\}$

$\mathcal{T}X = C_1 \cup C_2 \cup C_3$ **with**

$C_1 = \{(\xi, \xi - 2, \zeta) \mid \zeta \geq \xi - 1\}$

$C_2 = \{(\xi, \eta, \xi - 1) \mid \eta \geq \xi - 2\}$

$C_3 = \{(\xi, \eta, \eta + 1) \mid \xi \geq \eta + 2\}$

$\mathcal{T}\phi : (\alpha, \beta) \mapsto (\alpha + 1, \beta, \min\{\alpha, \beta + 1\})$

$\operatorname{im} \mathcal{T}\phi = C_2 \cup C_3$

Example, continued

$$\phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 & t^2 \\ 0 & 1 & -1 \\ 1 & t & 0 \end{bmatrix}$$

The last matrix contains all cocircuits of X, so

$$\operatorname{im} \mathcal{T}\left(\phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix}\right) = C_1 \cup C_2 \cup C_3 = \mathcal{T} X$$

by Yu and Yuster's theorem.

Rank two matrices

another example

$\phi : (\mathbb{A}^m)^2 \times (\mathbb{A}^n)^2 \to M_{m,n}, (x, y, z, u) \to xz^T + yu^T$

$X := \overline{\operatorname{im} \phi =}$ {rank-two matrices}

$X = \overline{\mathbb{T}^m Y \mathbb{T}^n}$, where

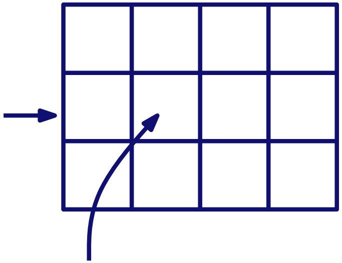
$Y := \{\mathbf{1}y^T + z\mathbf{1}^T\}$

$\rightsquigarrow \mathcal{T}X$ parameterised by

a co-circuit:

0	0	−1	0
1	1	0	1
0	0	−1	0

	ν_1	ν_2	ν_3	ν_4
ξ_1	γ_{11}	γ_{12}	γ_{13}	γ_{14}
ξ_2	γ_{21}	γ_{22}	γ_{23}	γ_{24}
ξ_3	γ_{31}	γ_{32}	γ_{33}	γ_{34}



$\min \gamma_{ij} + \xi_2 + \nu_2$

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Space of trees

parameterisation by splits

$\psi : \mathbb{A}^n \longrightarrow X \subseteq \mathbb{A}^{\binom{n}{2}}, \quad (x_1, \ldots, x_n) \mapsto (x_i - x_j)_{i<j}$

zero patterns in the image $\leftrightsquigarrow$ partitions of $[n]$

cocircuits $\leftrightsquigarrow$ partitions into two parts

$\rightsquigarrow \alpha : \mathbb{A}^{2^{n-1}-1} \longrightarrow \mathbb{A}^n$:

$\operatorname{im} \mathcal{T}(\psi \circ \alpha) = \mathcal{T}X = \mathcal{T}G_{2,n}$ up to lineality

Local tropical reparameterisations

Theorem

$\phi : \mathbb{A}^m \to \mathbb{A}^n$ polynomial map in characteristic zero
$X := \overline{\operatorname{im} \phi}$ algebraic variety of dimension d
Then $\exists \alpha : \mathbb{T}^d \to \mathbb{A}^m$ such that
$\dim \operatorname{im} \mathcal{T}(\phi \circ \alpha) = d (= \dim \mathcal{T} X)$.

proof sketch

1. $\mathcal{T} X$ pure d-dimensional complex, rationally defined over $v(K^*) \rightsquigarrow \exists \xi \in \mathcal{T} X$ such that $\dim(\langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{Q}} + v(K^*))/v(K^*) = d$; $\mu_1, \ldots, \mu_d \in \mathbb{R}$ a basis

Proof sketch, continued

2. $L := K(t_1, \ldots, t_d)$ with valuation $v(t_i) = \mu_i$
3. take a point p of $\mathbb{A}^m$ with coordinates in $\overline{\widehat{L}}$ such that $v(\phi(p)) = \xi$; exists
4. approximate p with $q \in K[t_1^{\pm 1/N}, \ldots, t_d^{\pm 1/N}]$ such that $v(\phi(q)) = \xi$ (multivariate Puiseux theorem)
5. set $u_i := t_i^{1/N}$
6. $q(u_1, \ldots, u_d)$ is the required reparameterisation; hits a d-dimensional neighbourhood of ξ.

Some remarks

- If all ϕ_i homogeneous of the same degree, k such local reparameterisations can be combined to a reparameterisation $\mathbb{A}^{kd} \longrightarrow \mathbb{A}^m$.
- Not yet very constructive, but I'm collaborating with Anders Jenssen to make it so.
- Not clear that finitely many suffice to cover $\mathcal{T}X$.