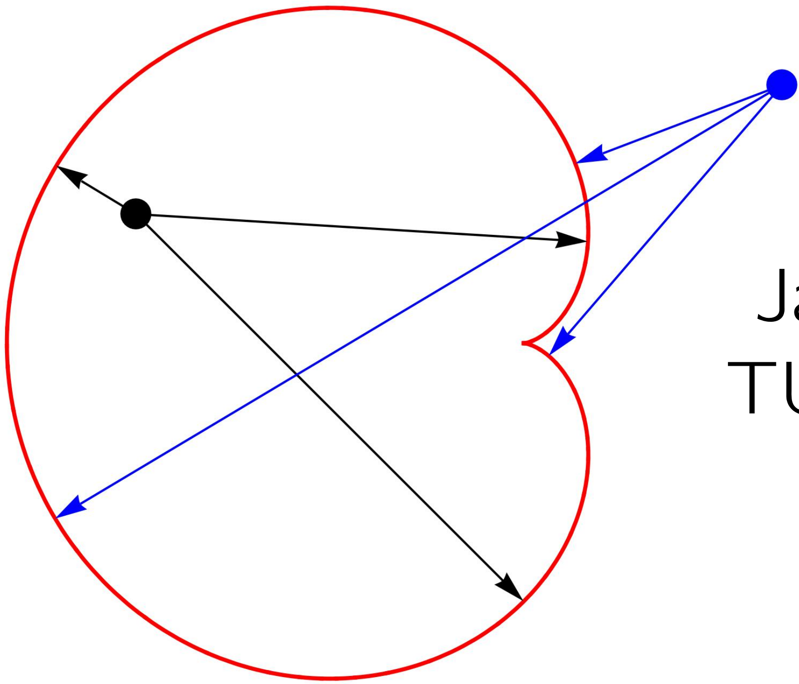
The Euclidean distance degree of an algebraic variety



Jan Draisma
TU Eindhoven

(w/ Horobeţ-Ottaviani-Sturmfels-Thomas) Basel, June 2014

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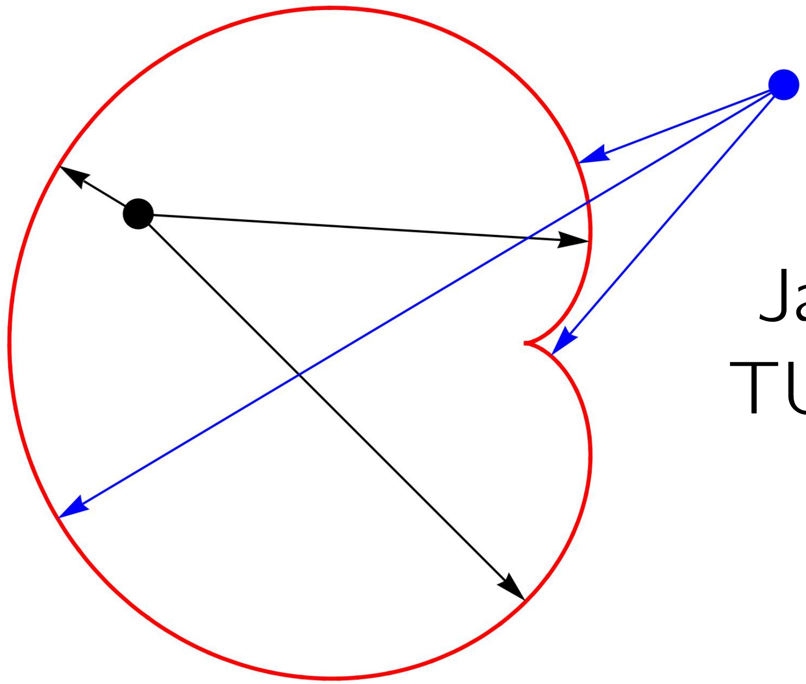

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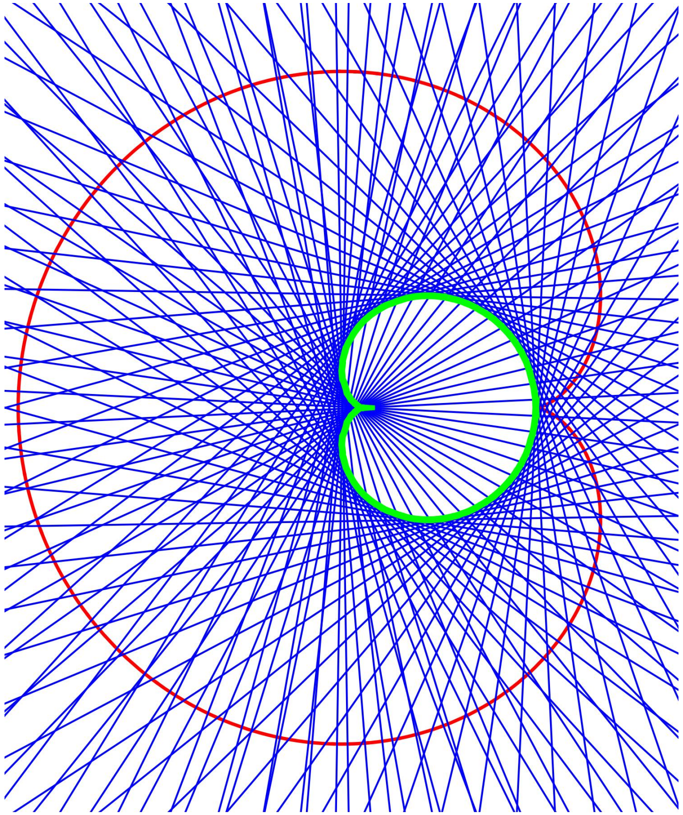

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ED degree

Definition

$V = \mathbb{C}^n$ with form $(.|.)$; $X \subseteq V$ subvariety; X_{reg} smooth locus; $u \in V$ general

$\rightsquigarrow$ $\mathrm{EDdegree}(X) := \#\{x \in X_{\mathrm{reg}} \mid u - x \perp T_x X\}$

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Goal

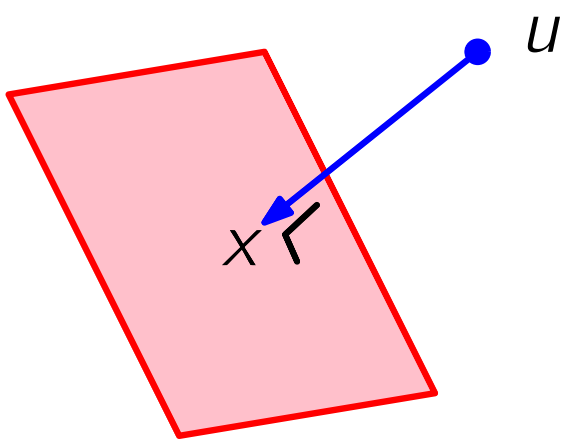
- techniques for computing $\mathrm{EDdegree}(X)$
- particularly for varieties from applications
- often $V = \mathbb{C} \otimes V_{\mathbb{R}}$, X complexification of $X_{\mathbb{R}} \subseteq V_{\mathbb{R}}$,

and also would like to count *real* critical points

Warmup: linear spaces

$X \subseteq V$ linear subspace

$\rightsquigarrow$ $\mathrm{EDdegree}(X) = 1$ if $X + X^{\perp} = V$; 0 otherwise



Warmup: bounded-rank matrices

$V = \mathbb{C}^{m \times n}$ with form $\mathrm{tr}(u^T v)$ and $m \geq n$
$X = \{\text{matrices of rank} \leq r\}$, stable under $\mathrm{O}_m \times \mathrm{O}_n$
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Eckart-Young

$$u = g \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & \\ \end{bmatrix} h$$

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compression by 90%

Example 1: Hurwitz stability

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$X_n := V(\det(H_n))$ *algebraic boundary*

n	3	4	5	6	7
EDdegree(X_n)	5	5	13	9	21

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n	3	4	5	6	7
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n	$2m+1$	$2m$
EDdegree(X_n)	$8m-3$	$4m-3$

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ED-correspondence

ED is degree additive $\rightsquigarrow$ may assume X is irreducible

$\mathcal{E}_X := \overline{\{(x,u) \in X_{\text{reg}} \times V \mid u - x \perp T_x X\}}$

$\rightsquigarrow$ *ED correspondence of* X

$\pi_1 : \mathcal{E}_X \to X$ affine bundle over X_{reg} of rank $c := n - \dim X$

$\pi_2 : \mathcal{E}_X \to \mathbb{C}^n$ has degree $\text{EDdegree}(X)$

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Lemma

If $T_xX \cap (T_xX)^\perp = \{0\}$ for some $x \in X_{\text{reg}}$,

then π_2 dominant and hence $\text{EDdegree}(X) > 0$.

(*Compute derivative of* π_2 *at* (x, x).)

Towards projective methods

$X \subseteq V$ irreducible affine cone $\rightsquigarrow \mathbb{P}X \subseteq \mathbb{P}V$ projective variety

$\mathbb{P}\mathcal{E}_X :=$ image of $\mathcal{E}_X$ under $X \setminus \{0\} \times V \to \mathbb{P}X \times V$

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$\pi_1 : \mathbb{P}\mathcal{E}_X \to \mathbb{P}X$ vector bundle over $\mathbb{P}X_{\text{reg}} \setminus Q$ of rank $c+1$

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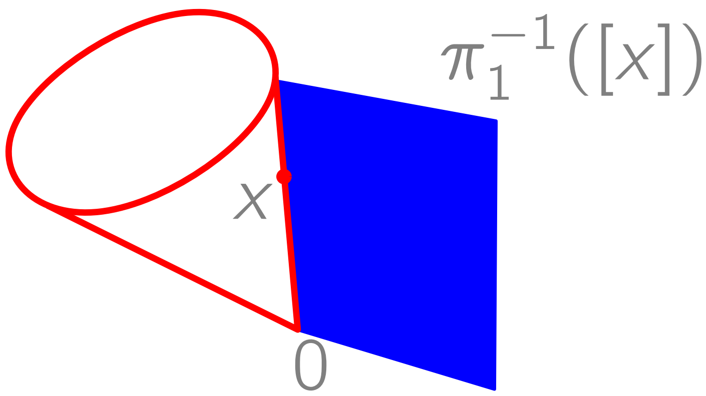

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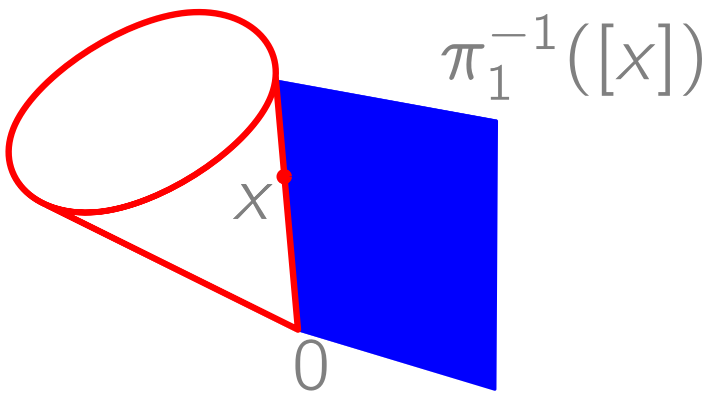
Observation

Point $u \in V$ gives section of $(\mathbb{P}X \times V)/\mathbb{P}\mathcal{E}_X$, and $\text{EDdegree}(X)$ counts zeroes of this section.



Duality

$X \subseteq V$ irreducible affine cone
$\rightsquigarrow \mathcal{N}_X := \overline{\{(x, y) \mid x \in X_{\mathrm{reg}}, y \perp T_x X\}} \subseteq V \times V$
conormal variety, $X^* := \overline{\pi_2(\mathcal{N}_X)}$ *dual variety*

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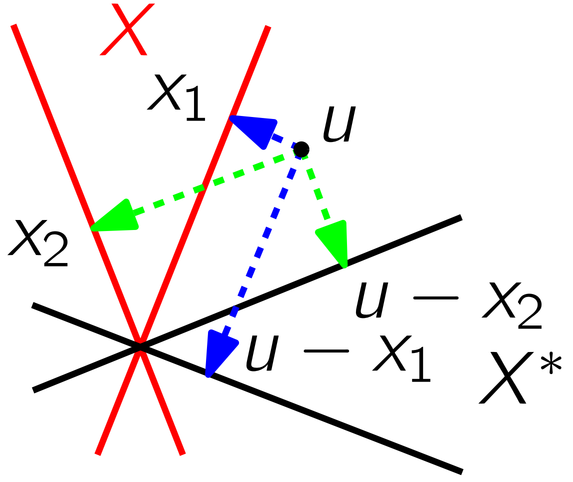
EDdegree(X) = EDdegree(X^*) (*see determinantal example!*)



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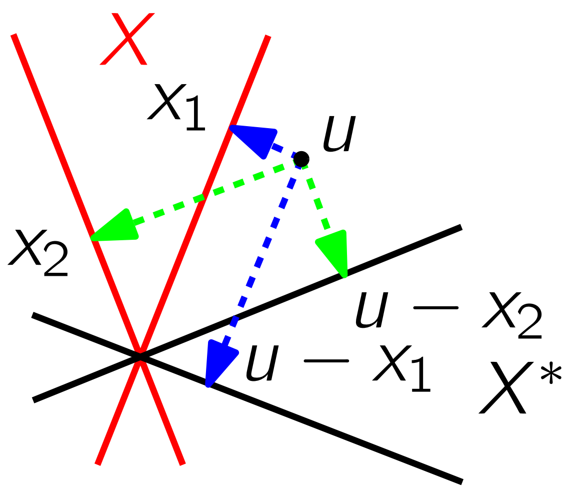

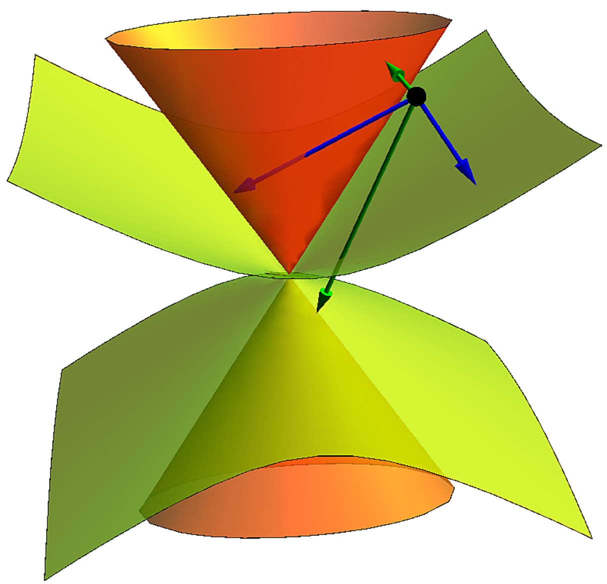

Polar classes

$X \subseteq V$ irreducible affine cone, $\dim(\mathbb{P}\mathcal{N}_X) = n-2 \rightsquigarrow$
$[\mathbb{P}\mathcal{N}_X] = \delta_0(X)s^{n-1}t + \delta_1(X)s^{n-2}t^2 + \ldots + \delta_{n-2}(X)s^1 t^{n-2}$
in cohomology ring $\mathbb{Z}[s,t]/(s^n,t^n)$ of $\mathbb{P}V \times \mathbb{P}V$

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If $\mathcal{N}_X$ does not intersect $\Delta \subseteq \mathbb{P}V \times \mathbb{P}V$, then
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Sufficient: $\mathbb{P}X \cap \mathbb{P}Q$ *is transversal and contained in* $\mathbb{P}X_{\mathrm{reg}}$.

Proof of polar class formula

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Proof

$Z \subseteq \mathbb{P}V \times \mathbb{P}V \times V$ variety of dependent triples $([x],[y],u)$

$(x,u) \in \mathcal{E}_X \rightsquigarrow ([x],[u-x],u) \in (\mathbb{P}\mathcal{N}_X \times V) \cap Z$

$([x],[y],u) \in (\mathbb{P}\mathcal{N}_X \times V) \cap Z \rightsquigarrow u = cx + dy \rightsquigarrow (cx, u) \in \mathcal{E}_X$

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Now $[Z_u] = s^{n-2} + s^{n-3}t + \cdots + st^{n-3} + t^{n-2}$ and

$\text{EDdegree}(X) = [Z_u] \cdot (\delta_0 s^{n-1}t + \cdots + \delta_{n-2}st^{n-1}) \bmod \langle s^n, t^n \rangle$

$\square$

Consequences of polar class formula

using **Holme 88**:

$\mathbb{P}X$ is smooth, m-dim, and transversal to $\mathbb{P}Q$ $\rightsquigarrow$

$\text{EDdegree}(X) = \sum_{i=0}^{m} (-1)^i \cdot (2^{m+1-i} - 1) \cdot \deg c_i(X)$

$c_i(X)$: i-th Chern class of the tangent bundle of X

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Toric case

$\mathbb{P}X$ smooth toric $+$ embedding

$\rightsquigarrow$ simple lattice polytope $P \subseteq \mathbb{R}^m$ with n lattice points

$\rightsquigarrow \deg c_i(X) = \sum$ normalised volumes of $(m-i)$-dim faces

Rational normal curves

$\mathbb{C}^2 \to S^n\mathbb{C}^2, w \mapsto w^n$; $X =$ image

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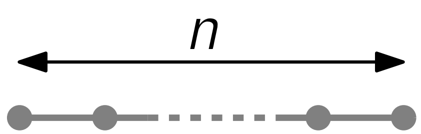
Polar class formula with $P =$ 

$\mathrm{EDdegree}(X) = (2^2 - 1) \cdot n - (2^1 - 1) \cdot 2 = 3n - 2$

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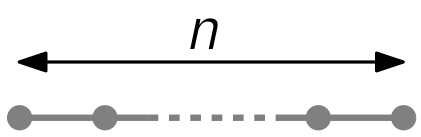
Polar class formula with $P =$ 

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If $(.|.)$ **is** O_2**-invariant**

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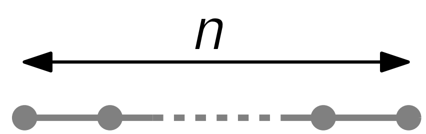
Polar class formula with $P =$ 

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Average ED degree over $\mathbb{R}$

Draw u from Gaussian corresponding to O_2-invariant $(.|.)$

$\rightsquigarrow$ expected $\#$ real critical points of d_u is $\sqrt{3n-2}$.

Example 2: Critical formations on a line

Motivation (Anderson-Helmke 2014)

Given $u_{ij} \in \mathbb{R}_{\geq 0}$ for $1 \leq i < j \leq p$, find $z \in \mathbb{R}^p$ that minimises $\sum_{i<j}(u_{ij} - (z_i - z_j)^2)^2$. More generally, # critical $z \in \mathbb{C}^p$?

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$X = \{((z_i - z_j)^2)_{i<j}\} \subseteq \mathbb{C}^{p(p-1)/2}$ *Cayley-Menger variety*

$$\rightsquigarrow \text{EDdegree}(X) = \begin{cases} \frac{3^{p-1}-1}{2} & \text{if } p \equiv 1, 2 \bmod 3 \\ \frac{3^{p-1}-1}{2} - \frac{p!}{3((p/3)!)^3} & \text{if } p \equiv 0 \bmod 3 \end{cases}$$

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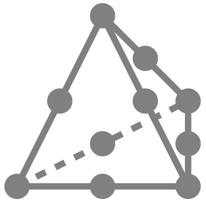

$\mathbb{P}X$ is 2nd Veronese embedding of $\mathbb{P}(\mathbb{C}^p/\mathbb{C}(1,\ldots,1))$

$\rightsquigarrow (m := (p-2))$-simplex P

$\rightsquigarrow \sum_{i=0}^{m}(-1)^{m-i} \cdot (2^{i+1} - 1) \cdot \boxed{\text{Vol}_i}$

Example 2, continued

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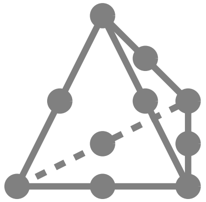

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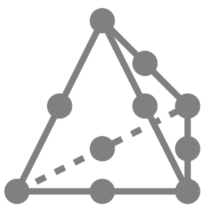
DHOST 2013

$X = \{((z_i - z_j)^2)_{i<j}\} \subseteq \mathbb{C}^{p(p-1)/2}$ *Cayley-Menger variety*

$\rightsquigarrow \text{EDdegree}(X) = \begin{cases} \frac{3^{p-1}-1}{2} & \text{if } p \equiv 1,2 \bmod 3 \\ \frac{3^{p-1}-1}{2} - \frac{p!}{3((p/3)!)^3} & \text{if } p \equiv 0 \bmod 3 \end{cases}$

Proof



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term 2 counts singularities of $\mathbb{P}X \cap \mathbb{P}Q$ $\square$

Back to example 1: Hurwitz stability

$$H_5 = \begin{bmatrix} x_1 & x_3 & x_5 & 0 & 0 \\ 1 & x_2 & x_4 & 0 & 0 \\ 0 & x_1 & x_3 & x_5 & 0 \\ 0 & 1 & x_2 & x_4 & 0 \\ 0 & 0 & x_1 & x_3 & x_5 \end{bmatrix}$$

$X_n := V(\det(H_n))$ *algebraic boundary*

DHOST 2013

n	$2m+1$	$2m$
EDdegree(X_n)	$8m-3$	$4m-3$

Example 3: Rank-one tensors

$V = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$, $(.|.)$ from standard forms on the factors

$X = \{v_1 \otimes \cdots \otimes v_p\}$, $\mathbb{P}X$ = Segre product $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}$

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Friedland-Ottaviani 2013

EDdegree(X) = coefficient of $s_1^{n_1-1} \cdots s_p^{n_p-1}$ in
$\prod_{i=1}^{p} \frac{\hat{s}_i^{n_i} - s_i^{n_i}}{\hat{s}_i - s_i}$, where $\hat{s}_i = s_1 + \cdots + s_{i-1} + s_{i+1} + \cdots + s_p$

Example 3, continued

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$([v],u)\in\mathbb{P}\mathcal{E}_X \Leftrightarrow \exists c\forall_i : u-cv\perp v_1\otimes\cdots\otimes V_i\otimes\cdots\otimes v_p$

$\Leftrightarrow u\perp v_1\otimes\cdots\otimes v_i^{\perp}\otimes\cdots\otimes v_p$ *(provided* $(v|v)\neq 0$*)*

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$\mathcal{E}_i$: bundle on $\mathbb{P}X$ with fibre $(v_1\otimes\cdots\otimes v_i^{\perp}\otimes\cdots\otimes v_p)^*$

$\rightsquigarrow u$ gives section of $\bigoplus_i\mathcal{E}_i$, top Chern class = coefficient. $\square$

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tensor format	EDdegree(X)	
$n_1 \times n_2$	$\min\{n_1, n_2\}$	(*Eckart-Young*)

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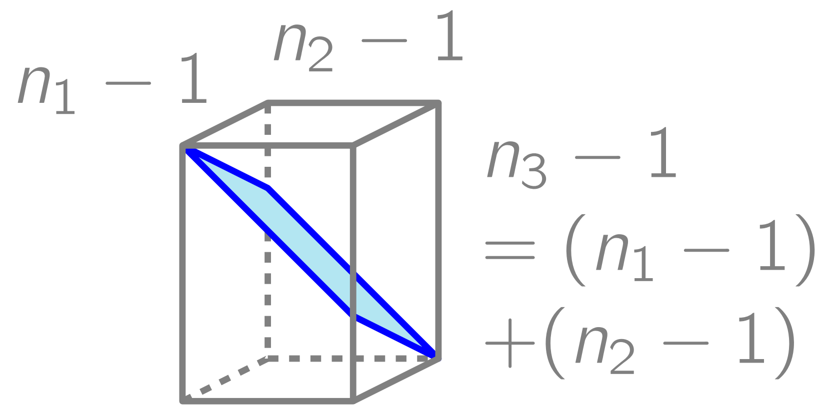

Boundary format (GKZ)

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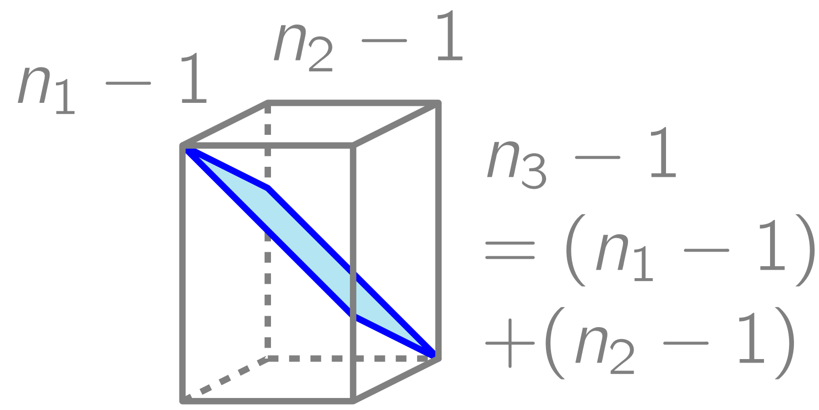

Boundary format (GKZ)

Intriguing problem

Find a *geometric* proof that

EDdegree(X) stabilises for $n_p - 1 \geq \sum_{i=1}^{p-1}(n_i - 1)$.

ED degs of groups (honours project Baaijens)

$W = \mathbb{C}^m$ with sym bilinear form, $V = \text{End}(W)$ with trace form

$G \subseteq V$ algebraic group $\rightsquigarrow$ EDdegree(G)?

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G preserves form on $W \rightsquigarrow x^T x \perp \mathfrak{g}$ so condition is $x^T u \perp \mathfrak{g}$
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Case 2

G does not preserve form

Closest orthogonal matrix

Folklore theorem

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Remarks

- used in computer vision for real u
- similar arguments work for unitary groups
- $\exists$ formula for the degree of a general G-orbit in $W^{\oplus m}$ (Kazarnovskiĭ, Derksen-Kraft), but $\mathfrak{g}^\perp$ not sufficiently general

Example 4: SL_n

Baaijens 2014

$\mathrm{EDdegree}(SL_m) = m! \cdot 2^{m-1}$ w.r.t. trace form

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- two papers: *arxiv: Euclidean distance degree*
- SIAM activity group in alg geom

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