Metric graphs with prescribed gonality

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Daejeon, August 2015

Metric graphs and harmonic maps

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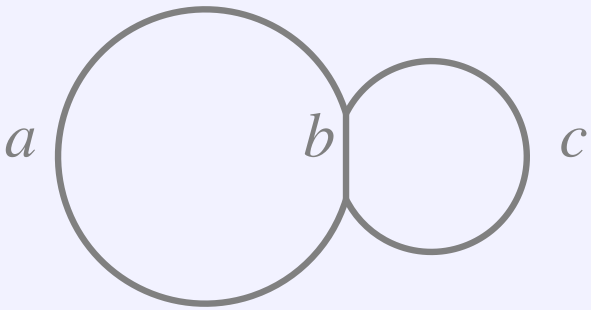

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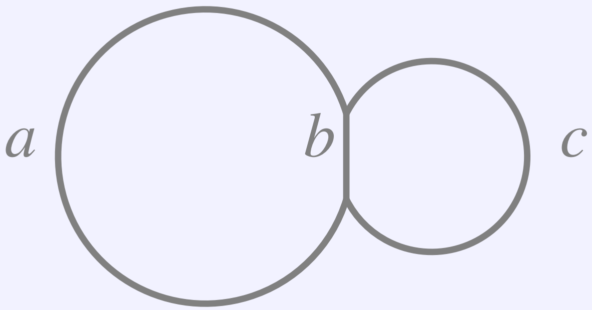

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Usually assumed connected; *genus*=g=first Betti number.

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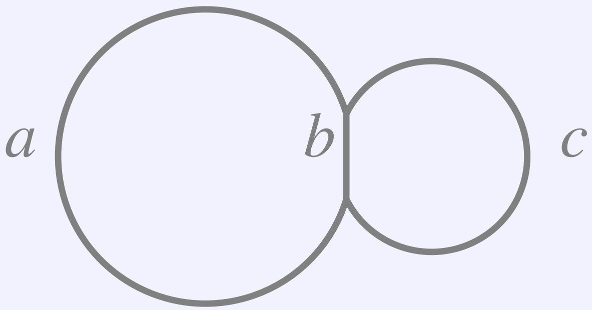

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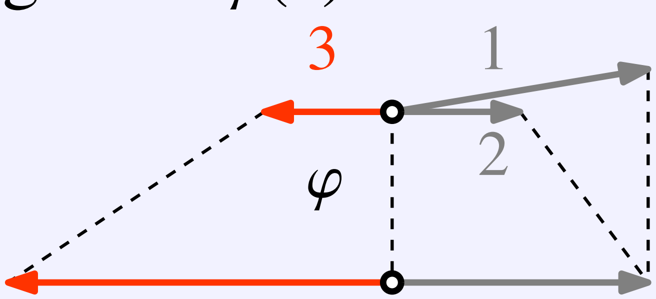

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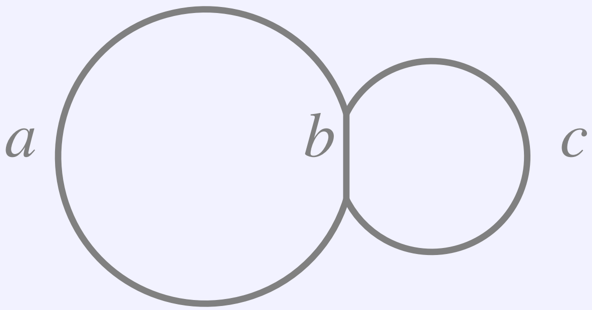

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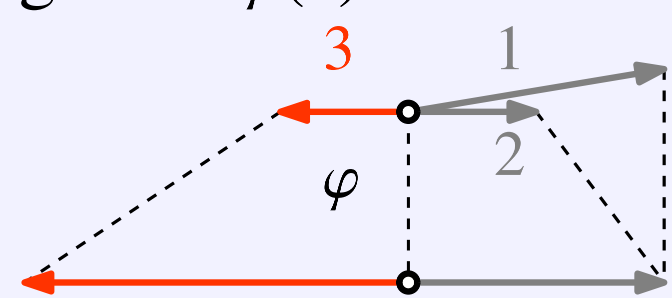

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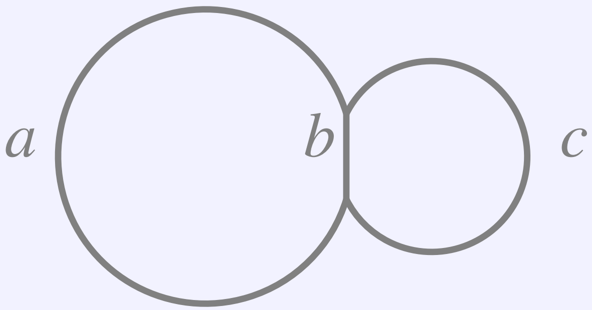

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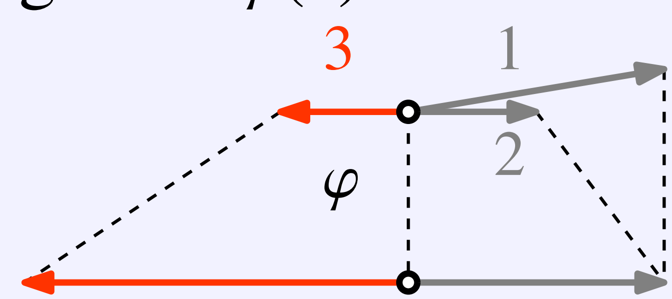

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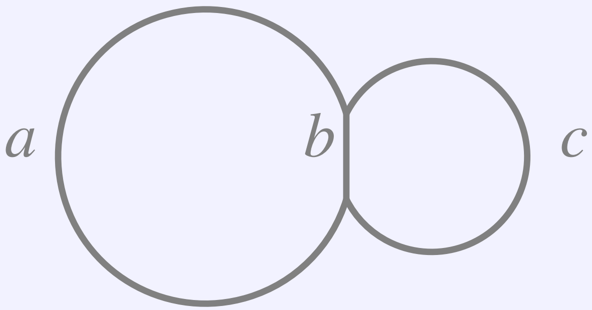

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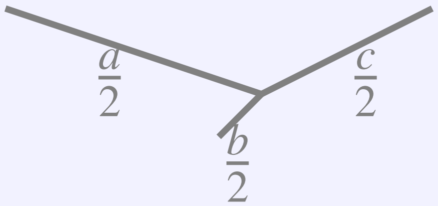

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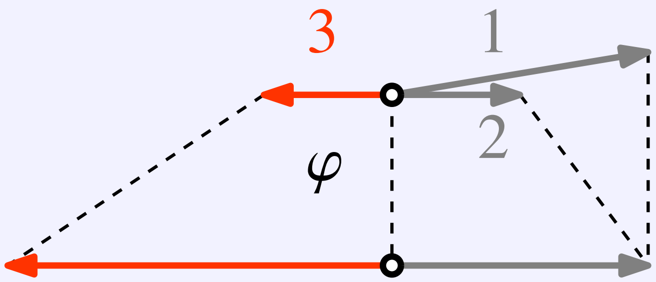

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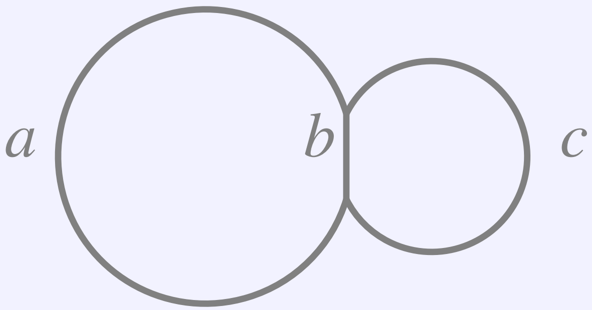

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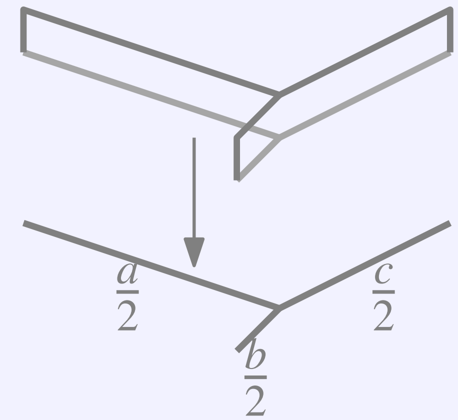

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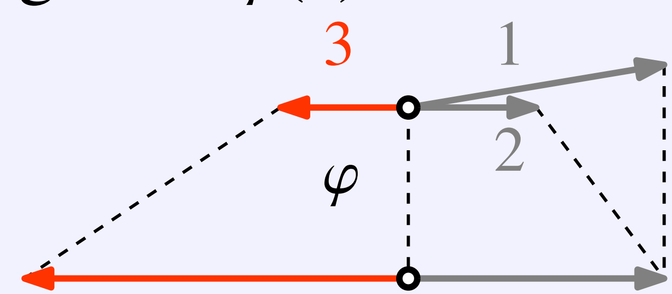

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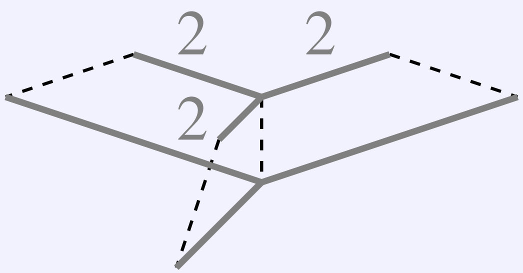

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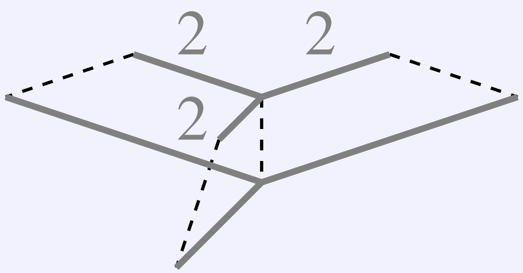

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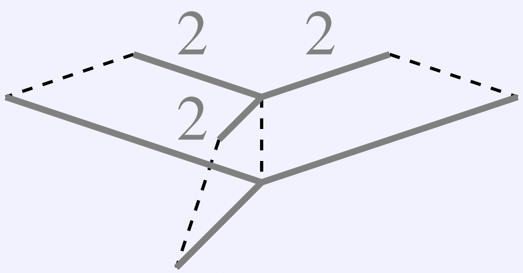

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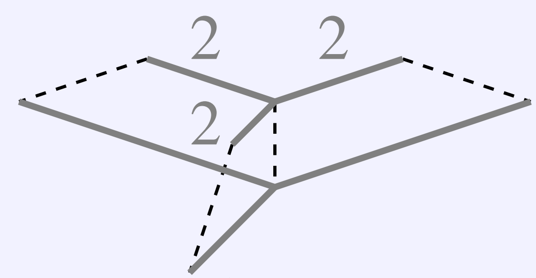

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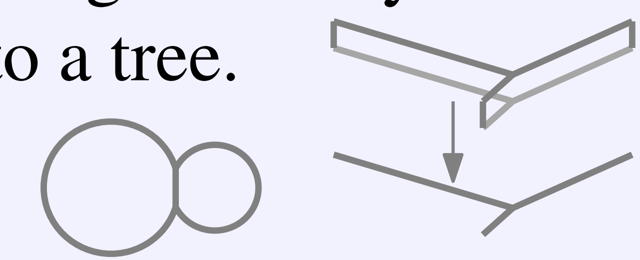
In the example, the gonality is 2*:*



Moduli space of genus-g metric graphs

- Let $g \geq 2$.
- For each ordinary genus-g graph $G = (V, E)$ set $C_G := (\mathbb{R}_{>0})^E$.
- For any isomorphism $G \to H$ glue C_G to C_H.
- If contracting e in G yields a genus-g graph H, glue C_H to C_G as the boundary with e-th coordinate 0.
- Identify modifications (ignore dangling trees).

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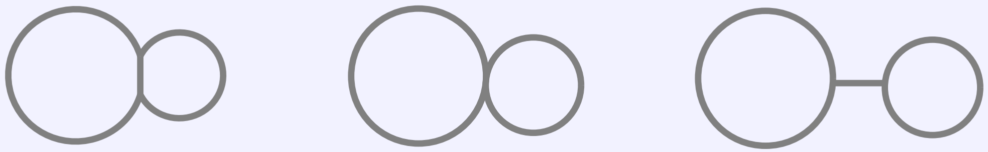
If G trivalent, then $\dim C_G = |E| = 3g - 3 \rightsquigarrow \dim M_g = 3g - 3$.



Theorem 1

For $d, g \geq 2$ the gonality-d locus in M_g is locally closed of dim $\min\{3g-3, 2g+2d-5\}$ (*perhaps not pure-dim*). In particular, the locus where the gonality is $\geq \lceil (g+2)/2 \rceil$ is dense and open.

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• Specialisation lemma and $\exists$ special divisors on curves $\Rightarrow$ the preimage in Thm 2 is *dense* in C_G. But our proof is combinatorial.
• Via the specialisation lemma, Theorem 1 implies that a general genus-g curve has gonality (at least) $\lceil (g+2)/2 \rceil$ — no need for a *specific* graph to prove this. (*Observed by Mikhalkin in 2011.*)

Gluing datum $(T, d, \sim)$

• Start with d copies $T_1, \ldots, T_d$ of a metric tree T.
• For each $v \in T$ let $\sim_v$ be an eq relation on $[d]$; this expresses which T_i are glued together at their respective copies of v.
• Assume that $\forall i, j : T_{ij} := \{v \in T \mid i \sim_v j\}$ is a finite union of disjoint closed *intervals* (or points).
• Let Γ be the glued object, with edges shrunk by a factor e where e edges are glued; assume connected.

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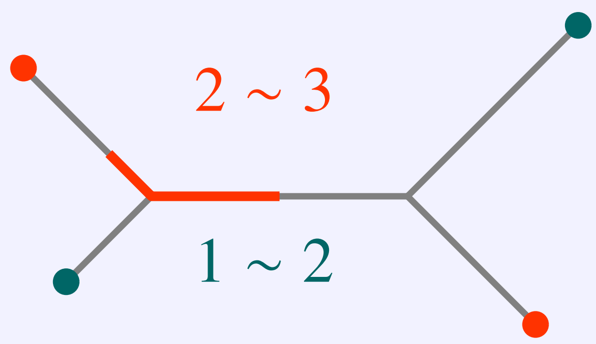

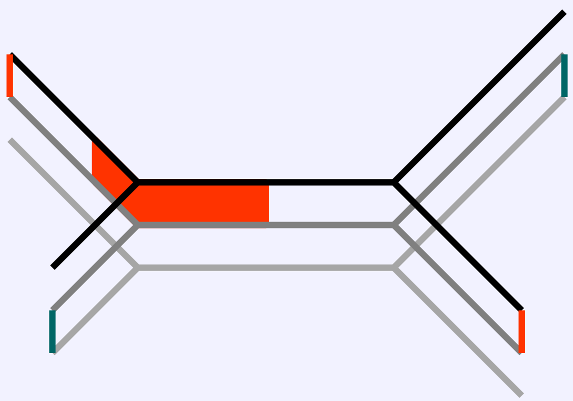

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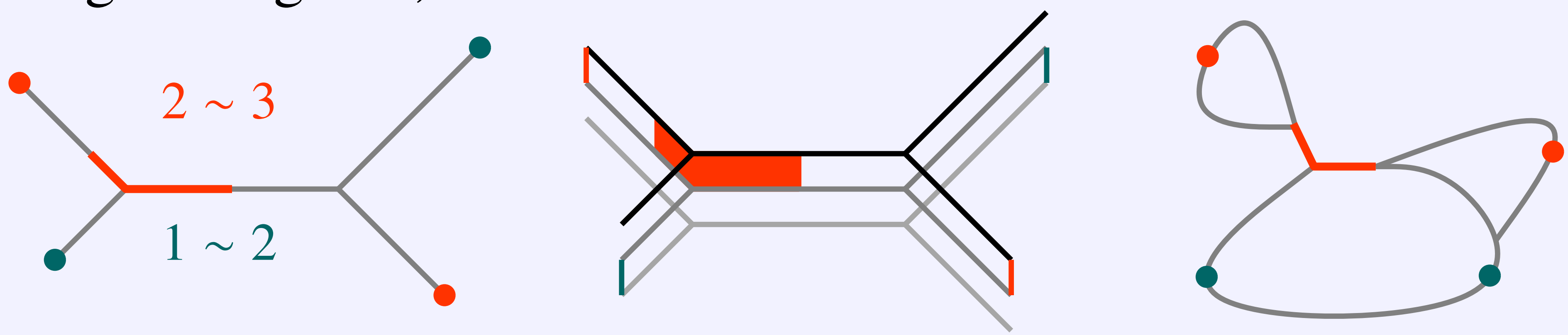

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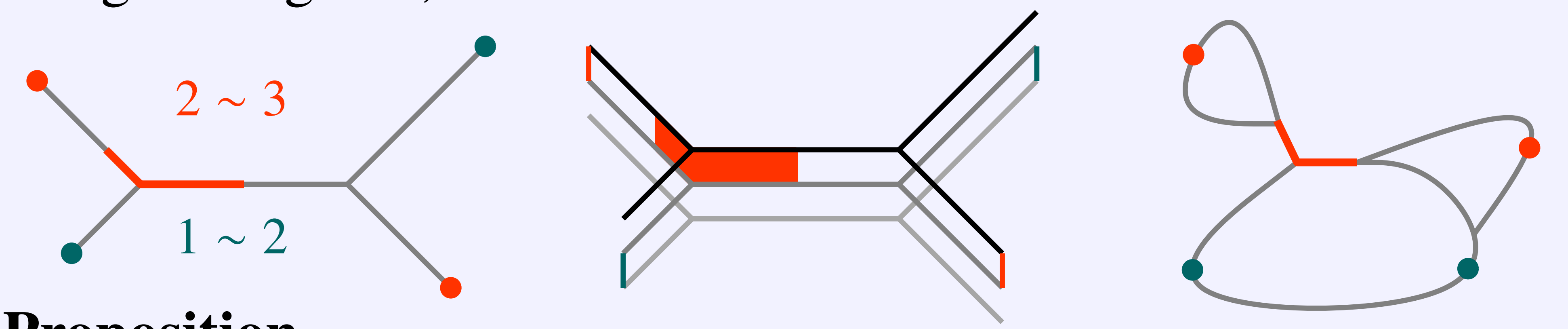

Proposition

The natural map $\varphi : \Gamma \to T$ is a degree-d tropical morphism, and all degree-d morphims to T arise in this manner.

Proposition

For $I \subseteq [d]$ set $T_I := \{v \in T \mid \forall i, j \in I : i \sim_v j\}$. The genus of Γ is $\sum_{I \subseteq [d]} (-1)^{|I|} c(T_I)$ where $c(.)$ counts the connected components.

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This turns out to imply that the edge lengths of only $2g + 2d - 5$ of the connected components of $T - E - \{\geq \text{trivalent vertices}\}$ contribute to the point in M_g represented by Γ. □

Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $\lceil (g+2)/2 \rceil =: d$ (Theorem 2).

• Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.

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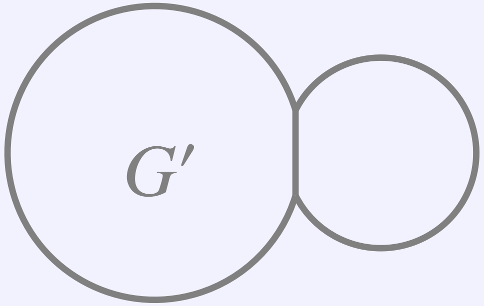

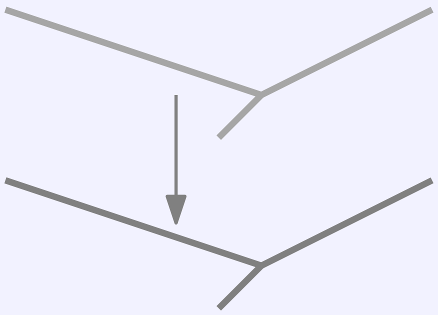

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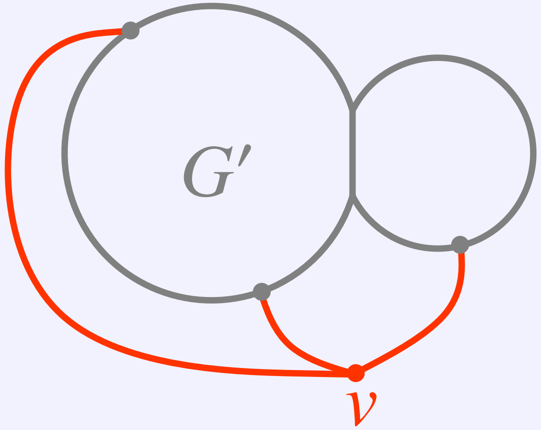

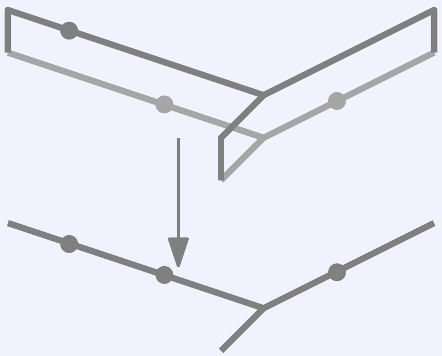

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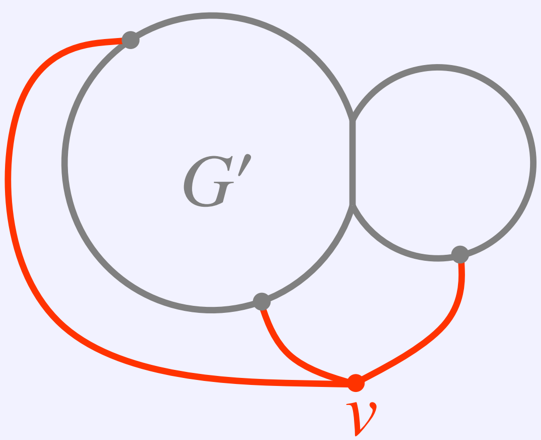

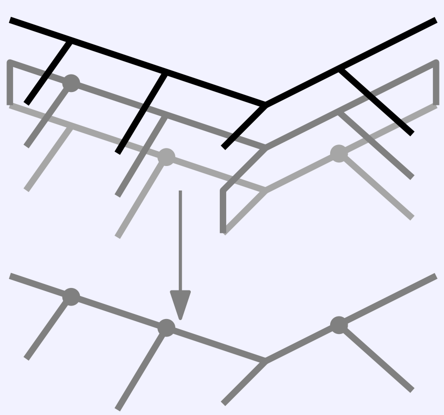

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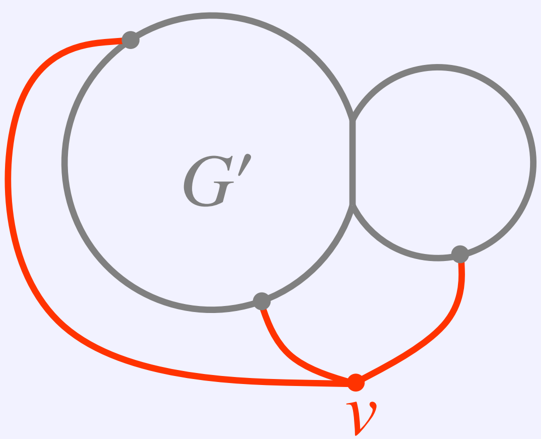

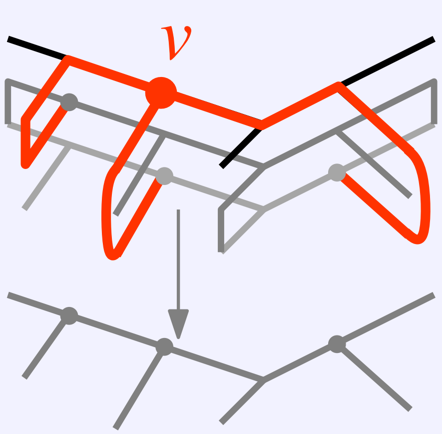

Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $\lceil (g+2)/2 \rceil =: d$ (Theorem 2).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
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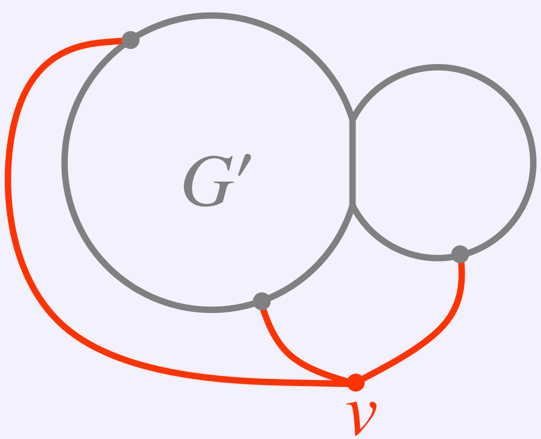

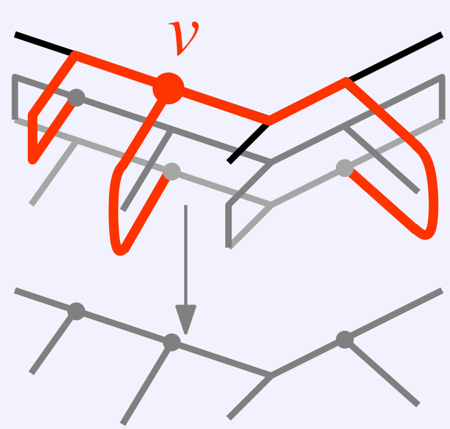

Parameter count:
3 for the gray dots
3 for the orange edges
$3g - 9 + 3 + 3 = 3g - 3$ $\square$

Concluding remarks

• If Γ in the open cone from theorem 2 has integral edge lengths, then it has a degree-$\lceil (g+2)/2 \rceil$ divisor of rank ≥ 1 supported on integral points.

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Thank you.