Learning $\underbrace{}_{\text{continuous}}$ symmetries of a variety

(Ongoing with a Bernoulli team, resp. joint with Biaggi-Gesmundo-Maraj-Mišinová.)

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June 2026

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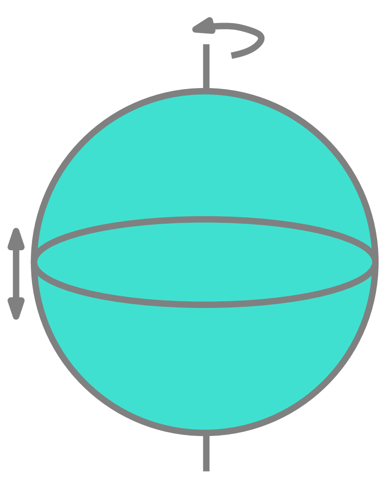
Running example
$X = S^2 \subseteq \mathbb{R}^3$
$G_X = \mathrm{O}_3 = \{\text{rotations and reflections}\}$,
$G_X^0 = \mathrm{SO}_3 = \{\text{rotations}\}$
$\mathfrak{g}_X = \mathfrak{so}_3 = \{A \mid A^T = -A\}$
$G_X / G_X^0 \cong \mathbb{Z}/2\mathbb{Z}$



Scenario 1: X given by equations

$I_X := \{f \in \mathbb{R}[y_1, ..., y_n] \mid f|_X \equiv 0\}$ vanishing ideal of X

GL_n acts on $\mathbb{R}[x_1, ..., x_n]$ by auto: $(gf)(p) = f(g^{-1}p)$

$\rightsquigarrow \mathfrak{gl}_n$ acts via derivations: $E_{ij} \cdot f = -y_j \frac{\partial f}{\partial y_i}$

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Easy fact: If I_X is generated by $f_1, ..., f_r$, then
$\mathfrak{g}_X = \{A \in \mathfrak{gl}_n \mid A \cdot I_X \subseteq I_X\} = \{A \mid \forall i : A \cdot f_i \in I_X\}$.

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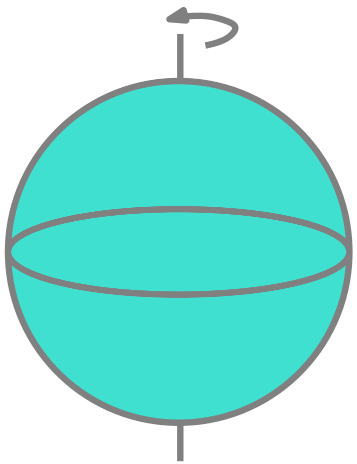
Algorithm 1 for $\mathfrak{g}_X$

On input generators $f_1, \ldots, f_r$ of I_X:

- Compute a Gröbner for I_X; let nf be the normal form map.
- Solve the linear system $\mathrm{nf}(A \cdot f_i) = 0,\ i = 1, \ldots, r$ for A.
- Return (a basis for) the solution space.

Running example

$$X = S^2 \subseteq \mathbb{R}^3 \rightsquigarrow I_X = \langle f := y_1^2 + y_2^2 + y_3^2 - 1 \rangle$$



$$A = \sum_{i,j} a_{ij} E_{ij} \rightsquigarrow A \cdot f = -2 \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} y_j y_i$$

$$\mathrm{nf}(A \cdot f) = -2 \sum_{i=1}^3 \sum_{j=1}^3 (a_{ij} - \delta_{ij} a_{11}) y_j y_i - 2a_{11}$$

$$\rightsquigarrow a_{11} = 0, a_{ii} = a_{11}, a_{ij} = -a_{ji} \rightsquigarrow A^T = -A$$

Interlude: testing binomiality

Assume X spans $\mathbb{R}^n$.

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[Kahle-Vill, 2024] derived an algorithm for testing whether I_X becomes binomial after a (potentially complex) linear change of coordinate:
• Compute $\mathfrak{g}_X$.
• Find a maximal toric subalgebra $\mathfrak{h}$ of $\mathfrak{g}_X$.
• If $\dim(\mathfrak{h}) < n$, then return "no", otherwise return $g \in \mathrm{GL}_n(\mathbb{C})$ that diagonalises $\mathfrak{h}$.

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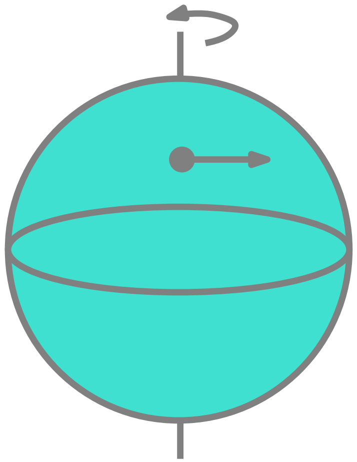
This works also in the following two scenarios!

Alternative characterisation of $\mathfrak{g}_X$

$A \in \mathfrak{g}_X \rightsquigarrow$ vector field on $\mathbb{R}^n$ tangent to X



Proposition: For any Zariski-dense $S \subseteq X$:
$\mathfrak{g}_X = \{A \in \mathfrak{gl}_n \mid \forall q \in S : A \cdot q \in T_q X\}$.

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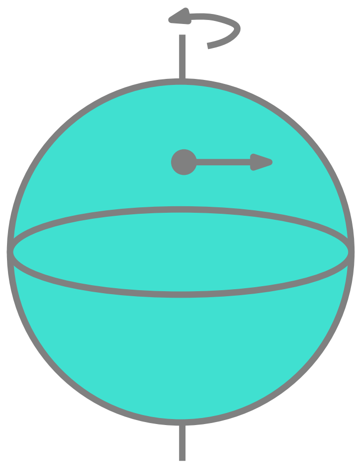

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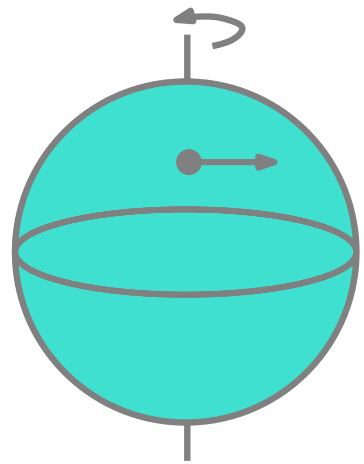

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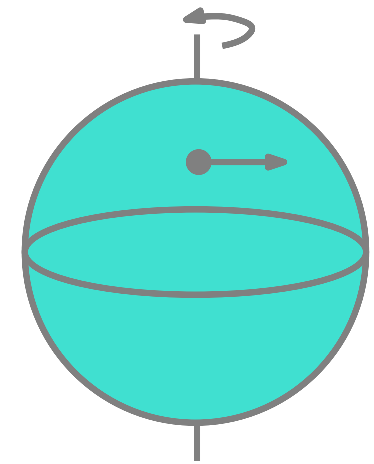

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Proof of $\supseteq$:
- $q \in \sigma_{r-1} Y$ and $p, p' \in Y$ sufficiently general, $A \in$ lhs.
- For sufficiently general $t \in \mathbb{R}$, $A \cdot (q + ty)$ lies in
$(T_q\sigma_{r-1} Y + T_y Y) \cap (T_q\sigma_{r-1} Y + T_{y'} Y) \underset{(*)}{=} T_q\sigma_{r-1} Y$
- Done by induction and proposition.

Scenario 2: parameterised X

$\varphi : \mathbb{R}^m \to \mathbb{R}^n$ polynomial (or rational) map defined over $\mathbb{Q}$:
$\varphi = (f_1, \ldots, f_n)$, $f_i \in \mathbb{Q}[x_1, \ldots, x_m]$ (or $\mathbb{Q}(x_1, \ldots, x_m)$).

$X := \overline{\mathrm{im}(\varphi)} \rightsquigarrow \mathfrak{g}_X = \langle \mathfrak{g}_X \cap \mathbb{Q}^{n \times n} \rangle =: \langle \mathfrak{g}_{X,\mathbb{Q}} \rangle$

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Remarks

- This naive algorithm is deterministic but not poly-time: compute generators $f_1, \ldots, f_r$ of I_X, then run Algorithm 1.
- Poly-time refers to bit complexity; and the f_i can even be given as an algebraic circuit.

Scenario 2: parameterised X

Algorithm 2 for $\mathfrak{g}_X$. Given $\varphi = (f_1, \ldots, f_n) \in \mathbb{Q}[x_1, \ldots, x_m]^n$.

- Pick random $p_1, p_2, \ldots, p_{n^2} \in \{1, \ldots, N(\epsilon)\}^m$;
- For each i, compute the column span T_i of $J(p_\ell) := (\frac{\partial f_i}{\partial x_j})$.
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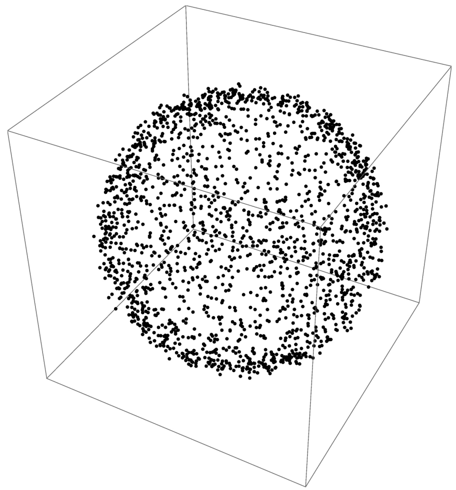
Remarks

- Output not guaranteed to contain or be contained in $\mathfrak{g}_X$.
- If $k = \dim(X)$ is known, can discard p_ℓ with rk$J(p_\ell) < k$ $\rightsquigarrow$ output will contain $\mathfrak{g}_X$, poly *expected* running time.
- In some applications, φ is given by a circuit—e.g., in undirected Gaussian graphical models, $\varphi(x) = x^{-1}$ for a symmetric matrix x with prescribed zero pattern—then use [Baur-Strassen, 1986].

Scenario 3: noisy samples

Goal:



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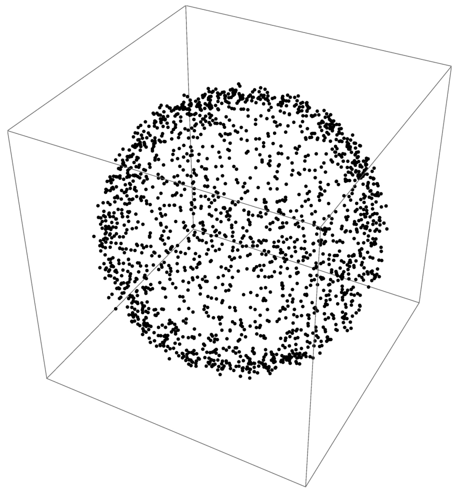

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Idea:

• From noisy samples compute pairs $(\widehat{q}_\ell, \widehat{T}_\ell)$ w.h.p. close to actual pairs $(q_\ell \in X_\ell, T_{q_\ell}X)$ (nearest neighbour, SVD).

• Solve $\forall \ell : A \cdot \widehat{q}_l \in \widehat{T}_\ell$ approximately using SVD, using sharp drop in singular values to decide on the dimension d of the solution space $\widehat{\mathfrak{g}}$.

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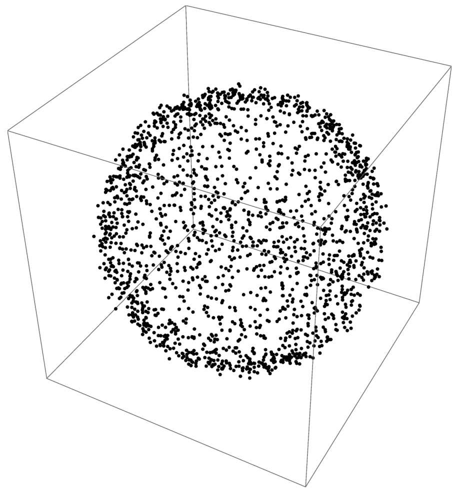

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Questions. Precise assumptions on X and the probability distribution? And how to *snap* $\widehat{\mathfrak{g}}$ to a *Lie algebra* near $\mathfrak{g}_X$?

The compact case

Assume X is compact.

Consequences:

• G_X is compact, and fixes an inner product β on $\mathbb{R}^n$.
• Estimate β via $\beta(A \cdot u, v) + \beta(u, A \cdot v) \approx 0$ for all u, v and $A \in \widehat{\mathfrak{g}}$ (again, SVD).
• After a linear coordinate change, β is the standard inner product $\rightsquigarrow$ may assume $\mathfrak{g}_X, \widehat{\mathfrak{g}} \in \mathrm{Gr}(d, \mathfrak{so}_n)$.

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This is a real algebraic subset, and we have to solve a near(est) point problem: given $\widehat{\mathfrak{g}}$, find $\mathfrak{g}_X \in \mathrm{Lie}(d, \mathfrak{so}_n)$.

- For $\mathfrak{g} \in$ Lie$(d, \mathfrak{so}_n)$ we have $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, contained in a maximal torus, and $\mathfrak{s}$ is semisimple.

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Proof: $T_{\mathfrak{g}}\mathrm{Gr}(d, \mathfrak{so}_n) = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^{\perp})$, and
$T_{\mathfrak{g}}\mathrm{Lie}(d, \mathfrak{so}_n) = \{\varphi : \mathfrak{g} \to \mathfrak{g}^{\perp} \mid \forall A, B \in \mathfrak{g} :$
$[A + \epsilon\varphi(A), B + \epsilon\varphi(B)] = [A, B] + \epsilon\varphi([A, B]) \bmod \epsilon^2\}$
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• Consequence: $\exists$ finitely many choices of $\mathfrak{s}$ up to conjugation by SO_n; these can be computed. [Dynkin, 1950s]

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- $V := \mathfrak{so}_n$ with inner product $(A|B) := -\mathrm{tr}(AB)$
- $\mathrm{Gr}(e, V) = \{P \in \mathrm{End}(V) \mid P^2 = P = P^T, \mathrm{tr}(P) = e\}$
- Induced metric is the *chordal metric*.

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• [Brockett, 1988] computed the corresponding gradient flow. In our case: $N' = [N, [N, P_{\widehat{\mathfrak{s}}}]]$ for $N = \mathrm{Ad}(g) P_{\mathfrak{s}_0} \mathrm{Ad}(g)^{-1}$.

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• Running this flow often $\rightsquigarrow$ Lie algebra $\mathfrak{s}$ close to $\widehat{\mathfrak{s}}$, hence to $\mathfrak{s}_X$. Finding $\mathfrak{z}$ close to $\mathfrak{z}_X$ in the centraliser of $\mathfrak{s}$ is easy.

Non-compact case

Many things become harder when X is not compact:

- Not sure what the output should look like. $\mathrm{Lie}(d, \mathfrak{so}_n)(\mathbb{Q})$ is Euclidean dense in $\mathrm{Lie}(d, \mathfrak{so}_n)$, but I don't know if that's true for $\mathfrak{gl}_n$ (probably not).

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