The Commutative Algebra of Highly Symmetric Data

Jan Draisma
TU Eindhoven

Utrecht, Nov 2013

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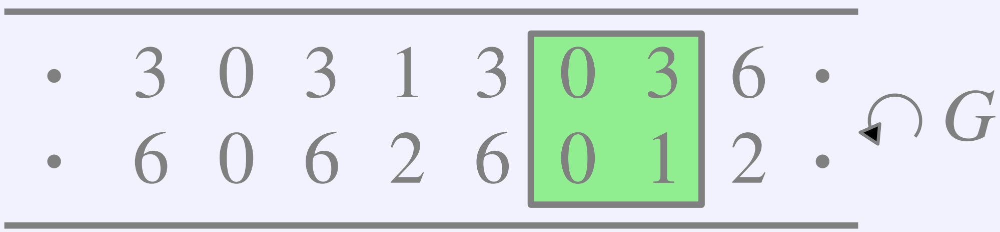
What global data properties can be tested locally?



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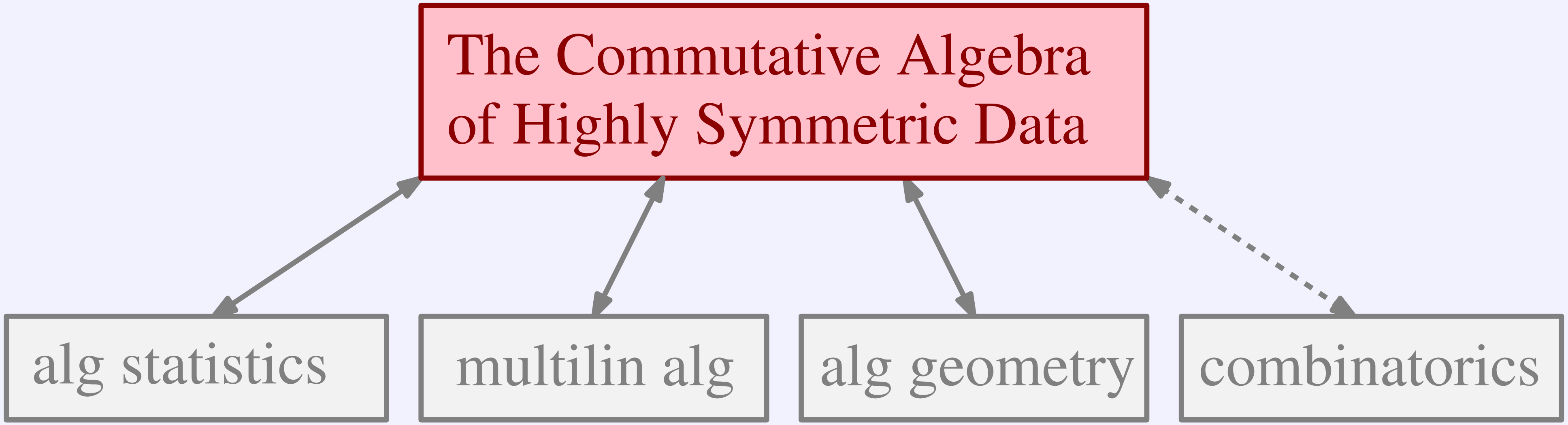

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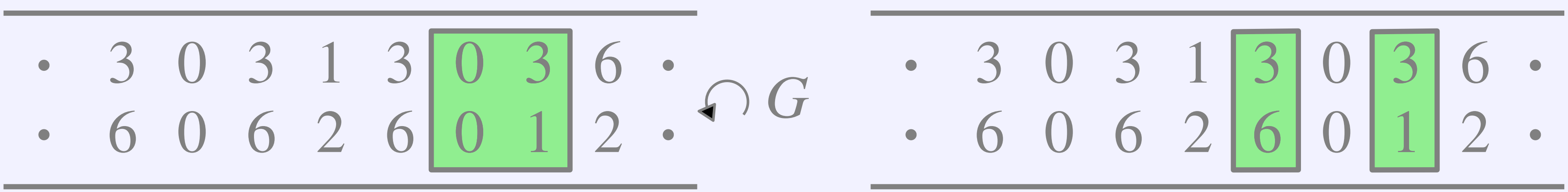

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History: Hilbert's Basis Theorem

David Hilbert

Any polynomial system

$f_1(x_1, \ldots, x_n) = 0,$

$f_2(x_1, \ldots, x_n) = 0, \ldots$

reduces to a *finite* system

($\rightsquigarrow$ *Noetherianity* of $K[x_1, \ldots, x_n]$)

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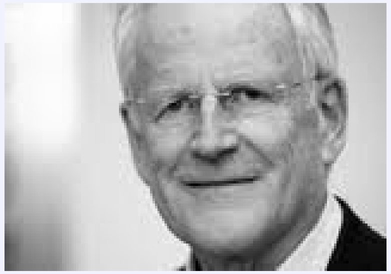
Bruno Buchberger
Gröbner bases, algorithmic methods



Three-step approach

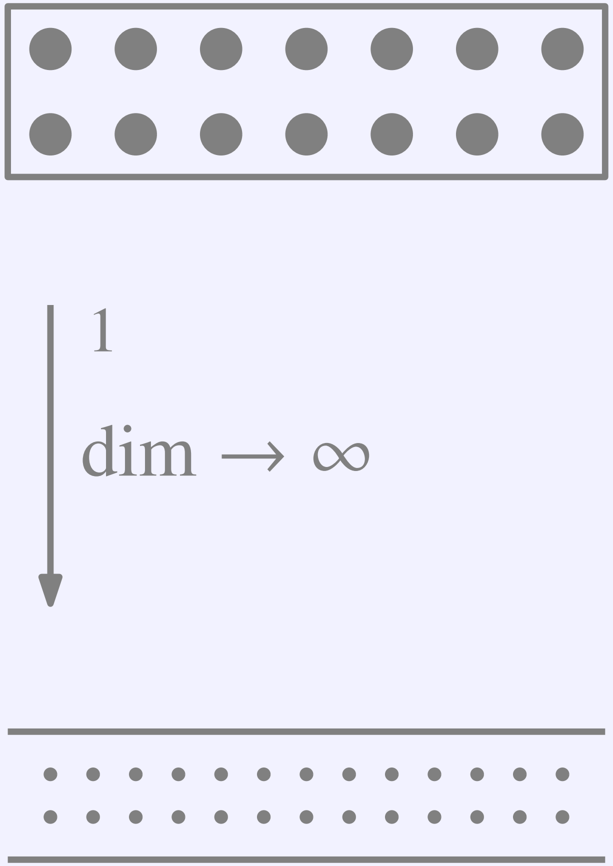
1. Model

high-dim data ⇝ ∞-dim data space

data property ⇝ ∞-dim subvariety

small window ⇝ finite window

⇝ *leave fin-dim commutative algebra*



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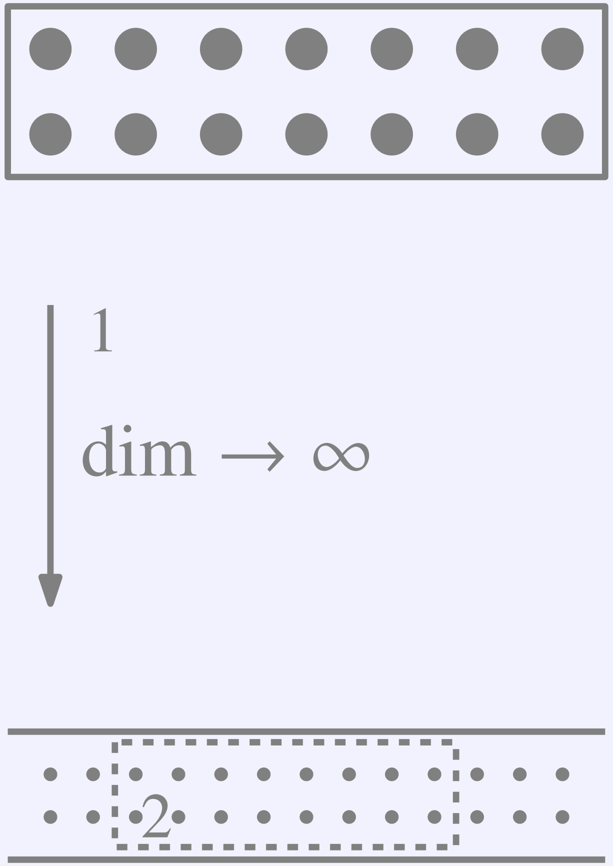
up to symmetry?
⇝ *generalise Basis Theorem to ∞ variables*



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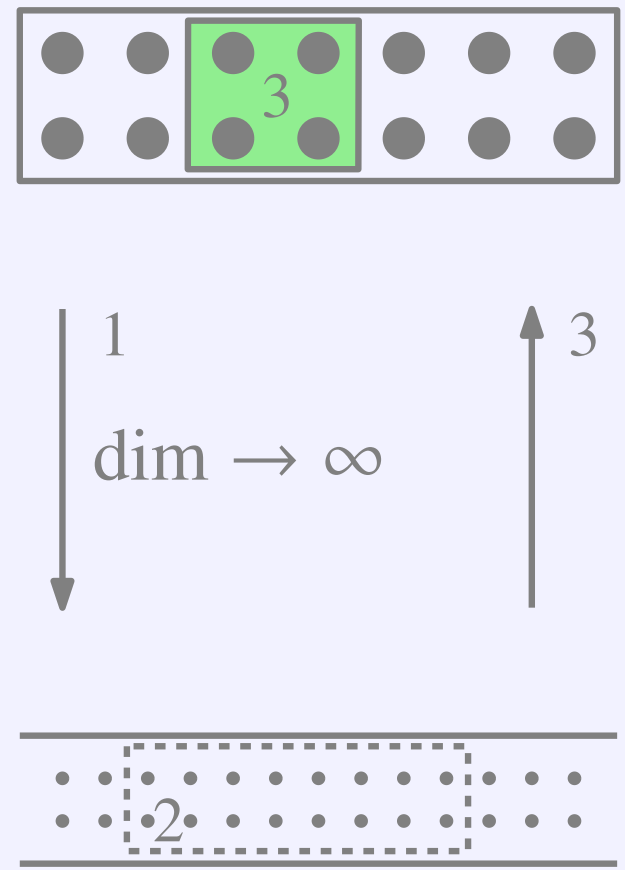
3. Compute

actual windows for fin-dim data

$\rightsquigarrow$ *generalise Buchberger alg to ∞ variables*



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$K[x_1, x_2, \ldots]$ is not Noetherian, e.g. $x_1 = 0, x_2 = 0, \ldots$ does not reduce to a finite system.

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Cohen [*J Algebra, 1967*]

$K[x_1, x_2, \ldots]$ *is* Sym($\mathbb{N}$)-Noetherian, i.e., every Sym($\mathbb{N}$)-stable system reduces to finitely many equations up to Sym($\mathbb{N}$).

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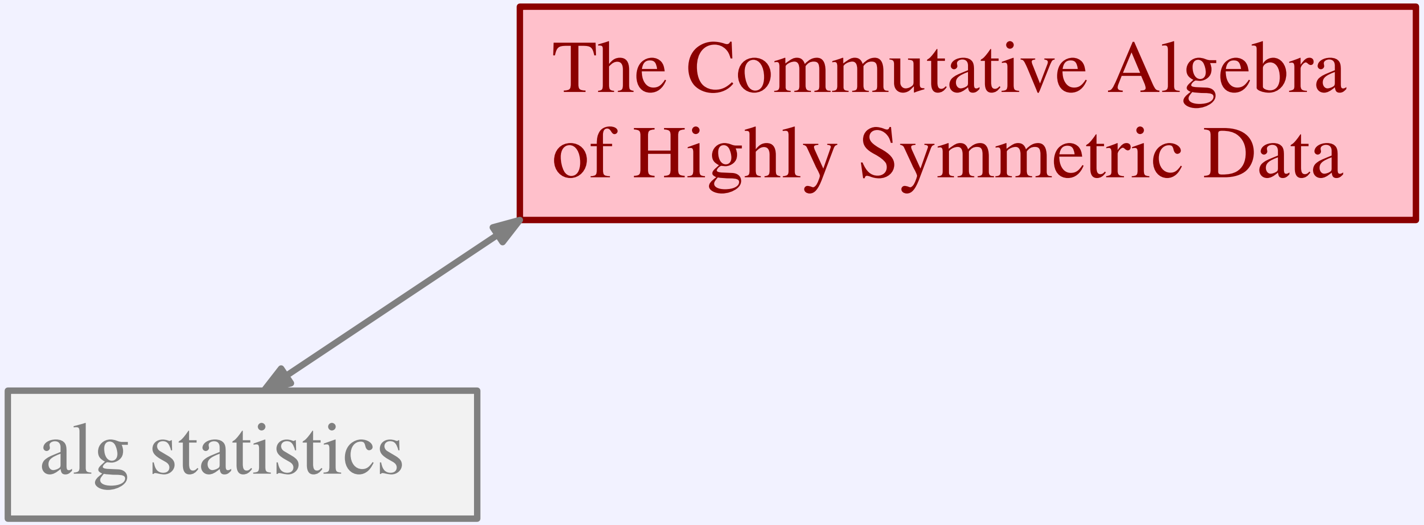
Eamples

$K[x_{ij} \mid i \in \{1, \ldots, k\}, j \in \mathbb{N}]$ is Sym($\mathbb{N}$)-Noetherian, but

$K[x_{ij} \mid i, j \in \mathbb{N}]$ is *not* Sym($\mathbb{N}$) × Sym($\mathbb{N}$)-Noetherian, but

$(K^{\mathbb{N}\times\mathbb{N}})^p$ with Zariski topology is $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian.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alg statistics

Setting

$X_1, \ldots, X_n$ jointly Gaussian, mean 0

$\rightsquigarrow$ explained well by $k \ll n$ *factors*?

i.e., is $X_i = \sum_j s_{ij} Z_k + t_i \epsilon_i$, with $Z_1, \ldots, Z_k$, $\epsilon_1, \ldots, \epsilon_n$ independent standard normals?

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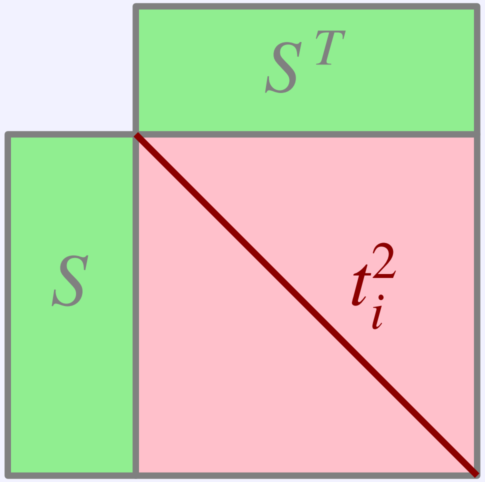
$\Leftrightarrow \Sigma(X_1, \ldots, X_n) = S\,S^T + \mathrm{diag}(t_1^2, \ldots, t_n^2)$



Gaussian factor analysis

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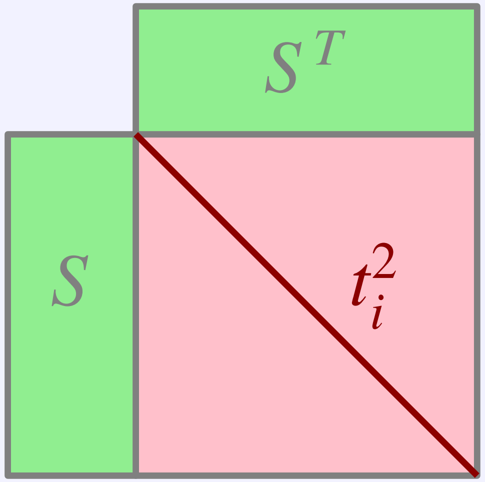

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Definition

$F_{k,n} := \overline{\{S S^T + \mathrm{diag}(t_1^2, \ldots, t_n^2) \mid S \in \mathbb{R}^{n \times k}, t_i \in \mathbb{R}\}}$

$\rightsquigarrow$ *algebraic variety in $\mathbb{R}^{n \times n}$ called Gaussian k-factor model*

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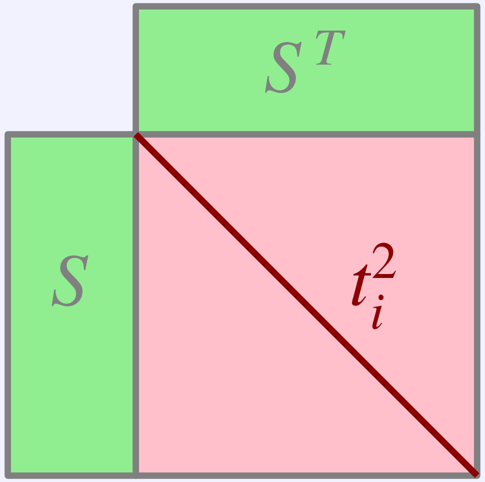

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Example

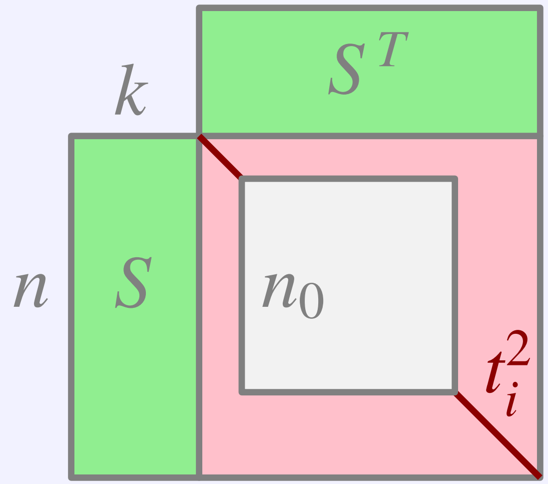
$F_{2,5}$ is zero set of $\{\sigma_{ij} - \sigma_{ji} \mid i, j = 1, \ldots, 5\}$ and the *pentad*
$\sum_{\pi \in \mathrm{Sym}(5)} \mathrm{sgn}(\pi) \sigma_{\pi(1)\pi(2)} \sigma_{\pi(2)\pi(3)} \sigma_{\pi(3)\pi(4)} \sigma_{\pi(4)\pi(5)} \sigma_{\pi(5)\pi(1)}$

Finiteness for the Gaussian k-factor model

Drton-Sturmfels-Sullivant [*Prob Th Rel Fields 2007*]

If $\Sigma \in F_{k,n}$ then any principal $n_0 \times n_0$ submatrix $\Sigma' \in F_{k,n_0}$.

$\rightsquigarrow$ *Is there an $n_0 = n_0(k)$ such that the converse holds for $n \geq n_0$?*



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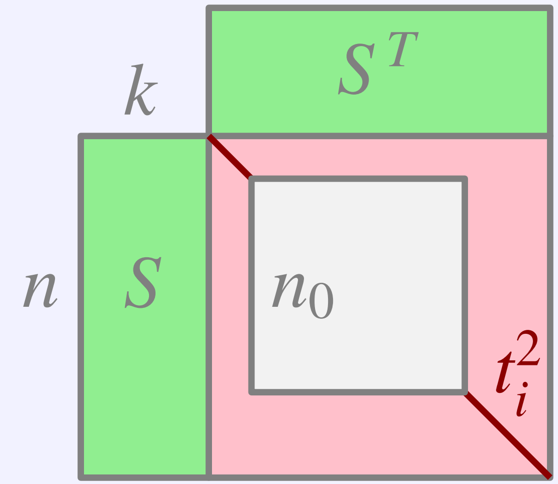

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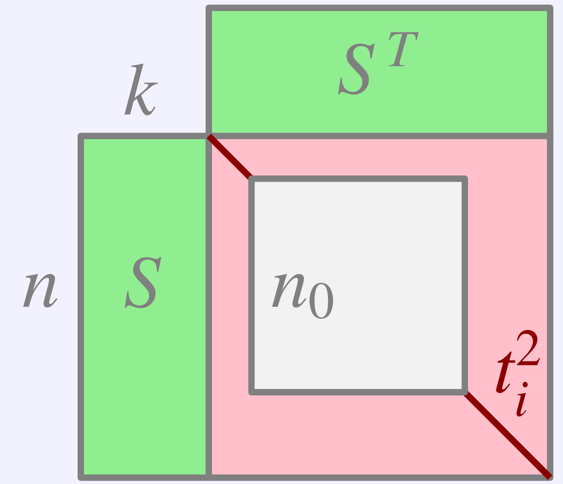
$\rightsquigarrow$ *uses $F_{k,\infty}$ and Noetherianity up to* $\mathrm{Sym}(\mathbb{N})$



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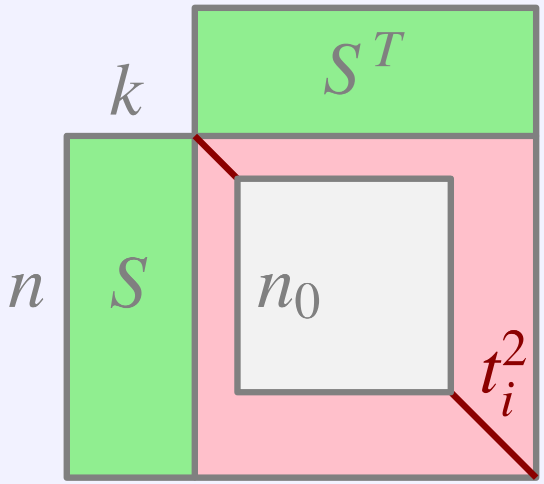

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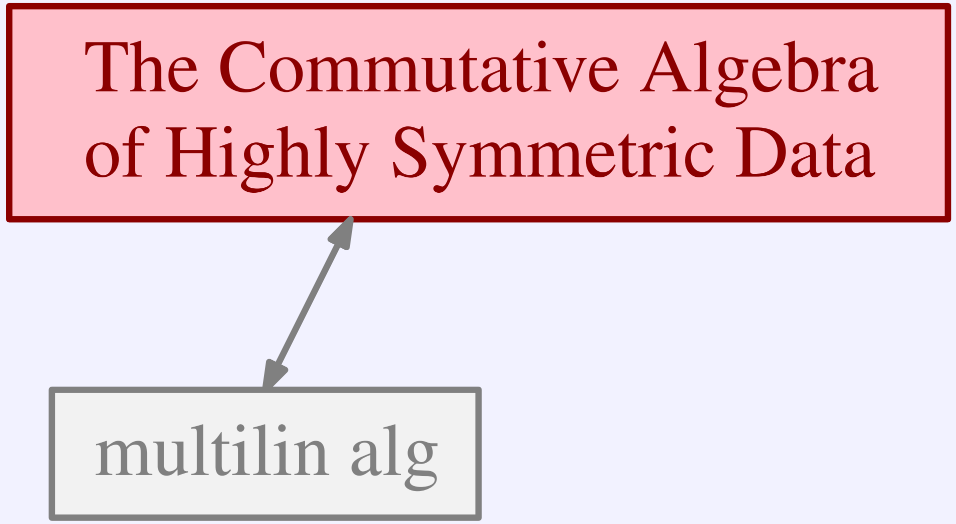
Brouwer-Draisma [*Math Comp 2011*]
yes for $k = 2$: pentads and 3×3-minors define $F_{2,n}$, $n \geq n_0 := 6$
$\rightsquigarrow$ *uses* Sym($\mathbb{N}$)*-Buchberger algorithm (+ a weekend on 20 computers)*
$\rightsquigarrow$ *a* **single** *computation proves this* **for all** n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multilin alg

A wrong-titled movie

tensor T=multi-indexed array of numbers

matrices=two-way tensors

this picture=three-way tensor, $\ldots$

Tensor rank

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Pure tensor P

has entries $P_{i,j,\dots,k} = x_i y_j \cdots z_k$

for vectors $x, \dots, z$

$\rightsquigarrow$ *for a matrix:* xy^T*, rank one*

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Tensor rank of T

is minimal k in $T = \sum_{j=1}^{k} P^{(j)}$ with each $P^{(j)}$ pure

$\rightsquigarrow$ *generalises matrix rank*

$\rightsquigarrow$ *useful for MRI data, communication complexity, phylogenetics etc.*

Finiteness for bounded-rank tensors

Matrix rank	Tensor rank
efficiently computable	NP-hard
field independent	field dependent
can only go down in limit	can also go up

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Draisma-Kuttler [*Duke 2014*]
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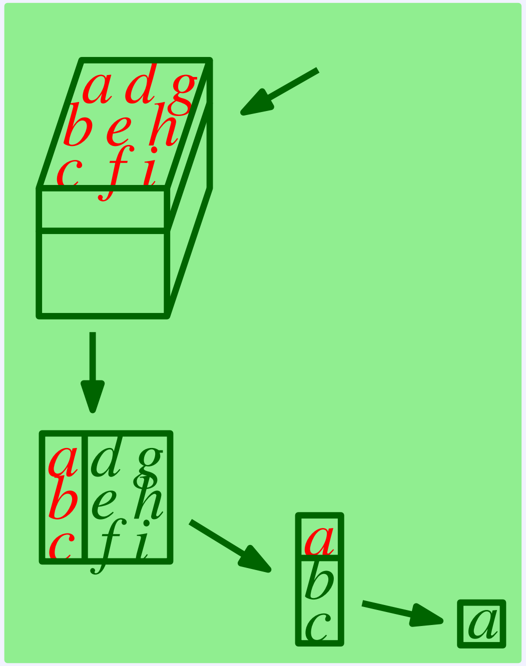

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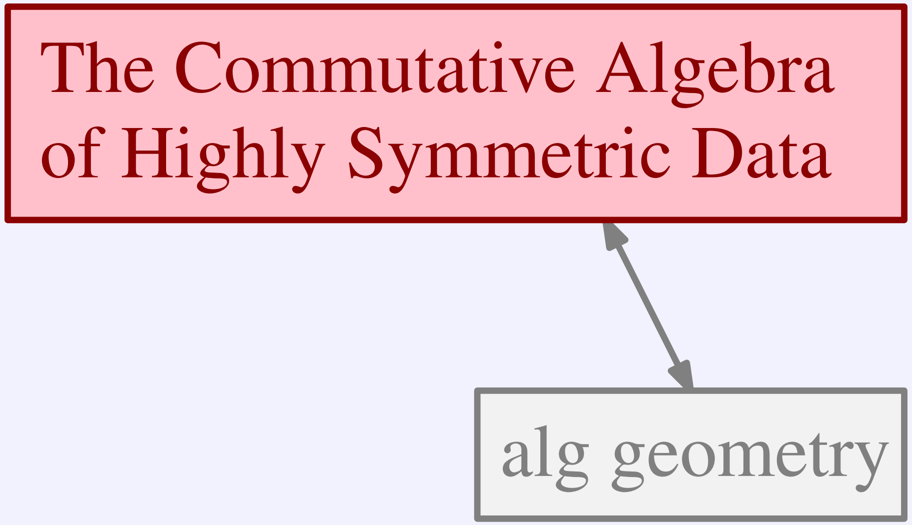
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alg geometry

Grassmannians: functoriality and duality

V a fin-dim vector space over an infinite field K
$\rightsquigarrow \mathbf{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \bigwedge^p V$
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Two properties:
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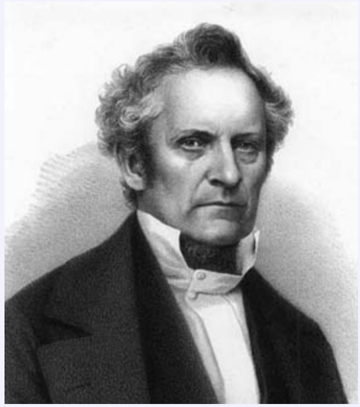
2. if $\dim V =: n + p$ with $n, p \geq 0$
$\rightsquigarrow$ natural map $\bigwedge^p V \to (\bigwedge^n V)^* \to \bigwedge^n(V^*)$
maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_n(V^*)$

Definition

Rules $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ with

$$\mathbf{X}_p : \{\text{vector spaces } V\} \to \{\text{varieties in } \textstyle\bigwedge^p V\}$$



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form a *Plücker variety* if, for $\dim V = n + p$,

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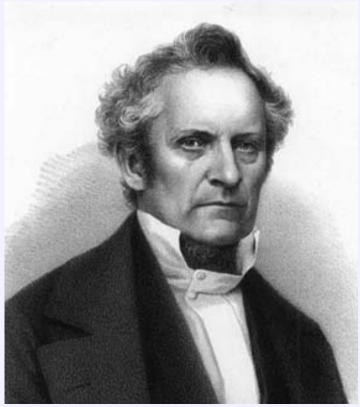
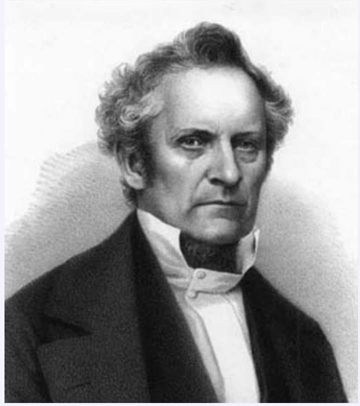


Plücker varieties



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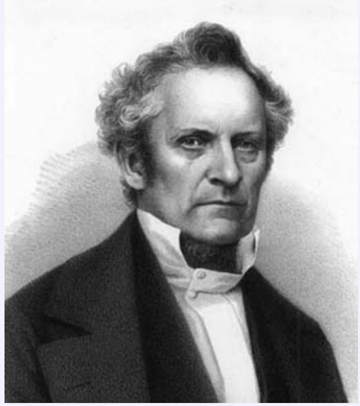
Constructions

$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are

$\mathbf{X} + \mathbf{Y}$ *(join)*, $\tau\mathbf{X}$ *(tangential)*,

$\mathbf{X} \cup \mathbf{Y}, \mathbf{X} \cap \mathbf{Y}$



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form a *Plücker variety* if, for $\dim V = n + p$,

1. $\varphi : V \to W \rightsquigarrow \bigwedge^p \varphi$ maps $\mathbf{X}_p(V) \to \mathbf{X}_p(W)$
2. $\bigwedge^p V \to \bigwedge^n (V^*)$ maps $\mathbf{X}_p(V) \to \mathbf{X}_n(V^*)$

Constructions
$\mathbf{X}, \mathbf{Y}$ Plücker varieties $\rightsquigarrow$ so are
$\mathbf{X} + \mathbf{Y}$ *(join)*, $\tau\mathbf{X}$ *(tangential)*,
$\mathbf{X} \cup \mathbf{Y}, \mathbf{X} \cap \mathbf{Y}$

skew analogue of Snowden's Δ*-varieties*

Definition
A Plücker variety $\{\mathbf{X}_p\}_p$ is *bounded* if $\mathbf{X}_2(V) \neq \bigwedge^2 V$ for $\dim V$ sufficiently large.

Finiteness for Plücker varieties, with Eggermont

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Theorem
For any fixed bounded Plücker variety there exists a polynomial-time membership test.

Theorems apply, in particular, to
$k\mathbf{Gr}$ = k-th *secant variety* of $\mathbf{Gr}$.

The infinite wedge

$$V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle_K$$
$$V_{n,p} := \langle x_{-n}, \ldots, x_{-1}, x_1, \ldots, x_p \rangle \subseteq V_\infty$$

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Diagram

$$\begin{array}{ccc}
\bigwedge^0 V_{00} & \bigwedge^1 V_{01} & \bigwedge^2 V_{02} \\
\downarrow & \downarrow & \downarrow \\
\bigwedge^0 V_{10} & \bigwedge^1 V_{11} & \bigwedge^2 V_{12} \\
\downarrow & \downarrow & \downarrow
\end{array}$$

$$\begin{array}{cc}
\bigwedge^p V_{n,p} & \bigwedge^{p+1} V_{n,p+1} \\
\downarrow & \\
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Definition

$\bigwedge^{\infty/2} V_\infty := \lim_{\rightarrow} \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \dots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

Young diagrams

Recall

$\bigwedge^{\infty/2} V_\infty$ has basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, where
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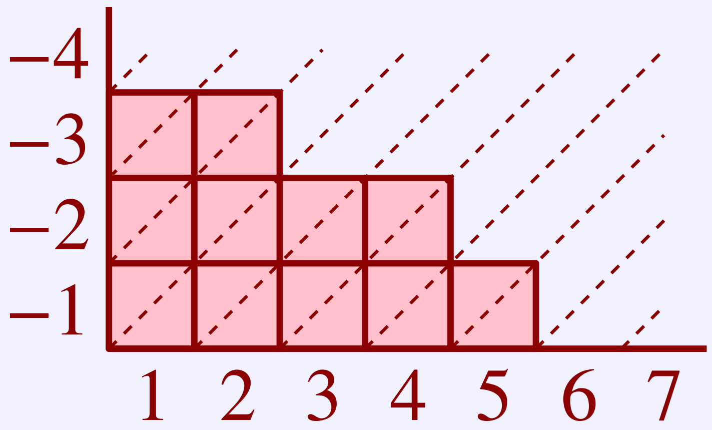
Bijection with Young diagrams

x_I with $I = \{-3, -2, 1, 2, 4, 6, 7, \ldots\}$ corresponds to



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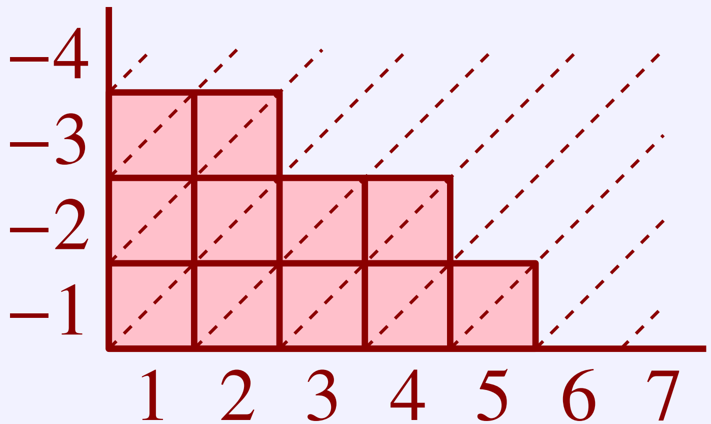

These x_I will be the *coordinates* of our ambient space, partially ordered by $I \leq J$ if $i_k \geq j_k$ for all k (inclusion of Young diags).
Unique minimum is $I = \{1, 2, \ldots\}$.

The limit of a Plücker variety

Dual diagram

$$
\begin{array}{ccccc}
\bigwedge^0 V_{00}^* & \longleftarrow & \bigwedge^1 V_{01}^* & \longleftarrow & \\
\uparrow & & \uparrow & & \\
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$\{\mathbf{X}_p\}_{p\geq 0}$ a Plücker variety $\rightsquigarrow$ varieties $X_{n,p} := \mathbf{X}_p(V_{n,p}^*)$

Dual diagram

$$\begin{array}{ccccc}
\bigwedge^0 V^*_{00} & \longleftarrow & \bigwedge^1 V^*_{01} & \longleftarrow & \\
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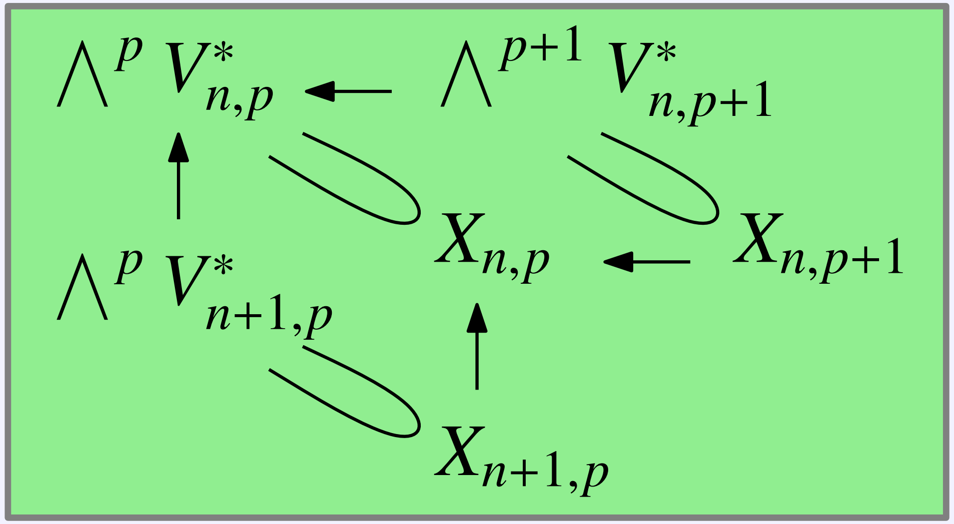

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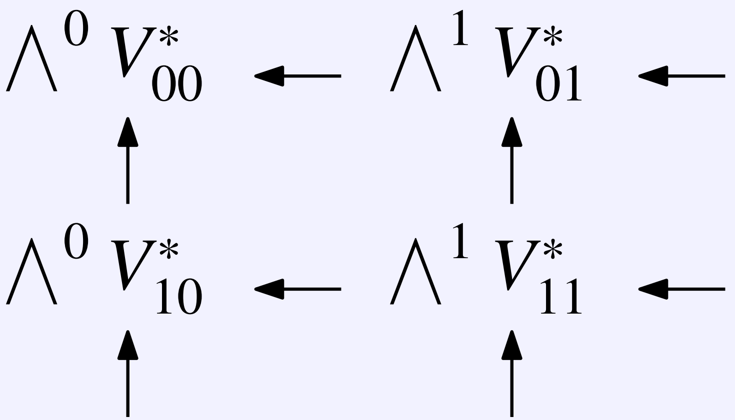

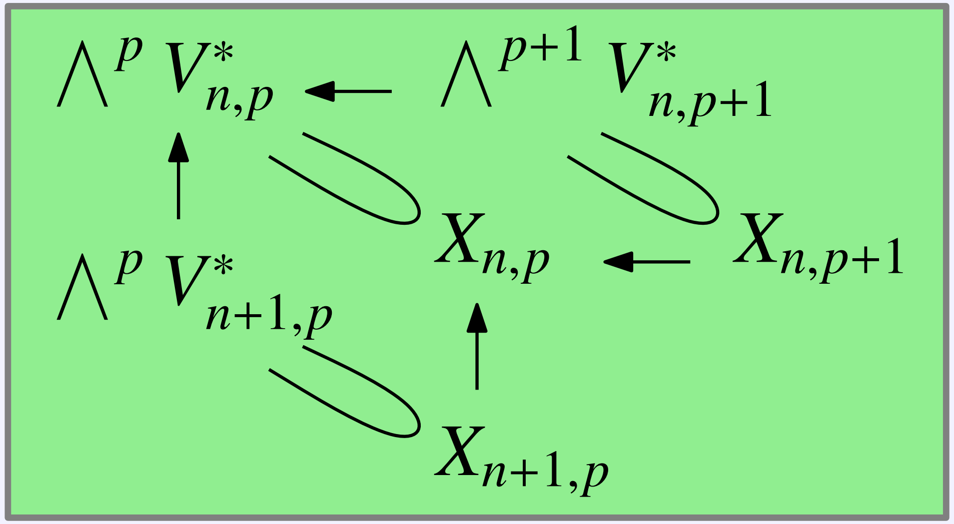

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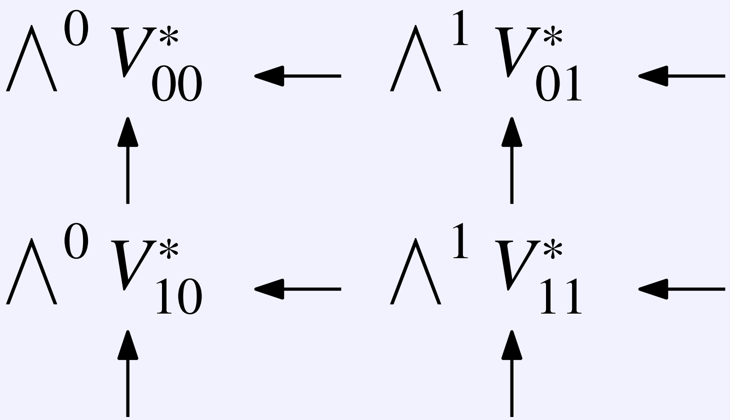

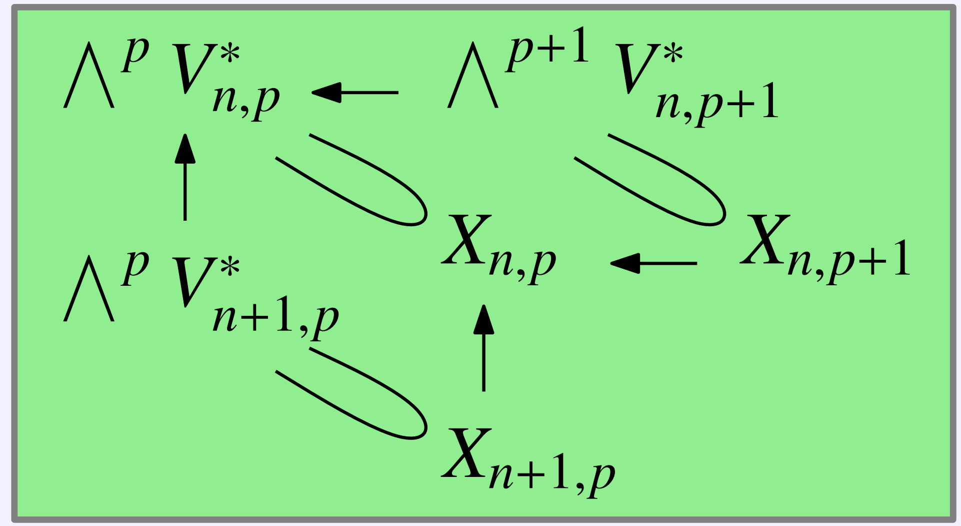

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Theorem (implies earlier)
For *bounded* $\mathbf{X}$, the limit $\mathbf{X}_\infty$ is cut out by finitely many GL_∞-orbits of equations.

Example

The limit $\mathbf{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$ of $(\mathbf{Gr}_p)_p$ is *Sato's Grassmannian* defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$ where $i_k = k - 1$ for $k \gg 0$ and $j_k = k + 1$ for $k \gg 0$.

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$\rightsquigarrow$ *not finitely many* GL_∞*-orbits*

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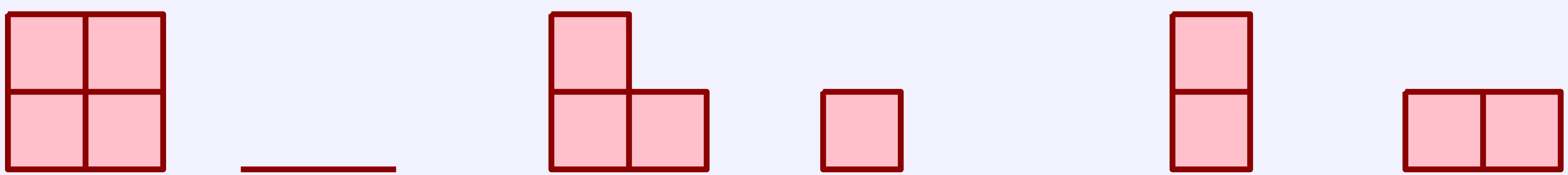
But in fact the GL_∞-orbit of

$(x_{-2,-1,3,\ldots} \cdot x_{1,2,3,\ldots}) - (x_{-2,1,3,\ldots} \cdot x_{-1,2,3,\ldots}) + (x_{-2,2,3,\ldots} \cdot x_{-1,1,3,\ldots})$



defines $\mathbf{Gr}_\infty$ set-theoretically.

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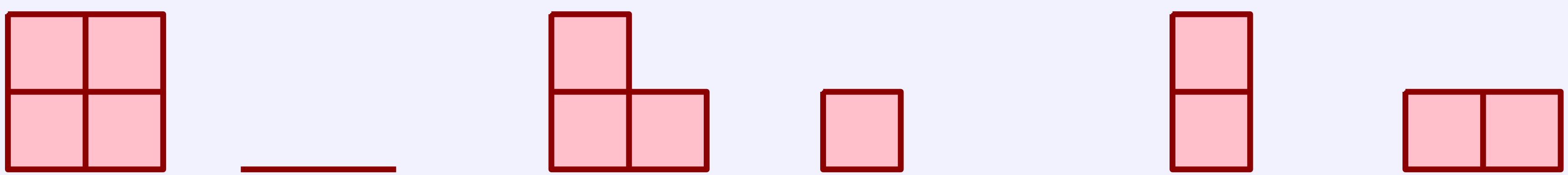


defines $\mathbf{Gr}_\infty$ set-theoretically.

Our theorems imply that also higher secant varieties of Sato's Grassmannian are defined by finitely many GL_∞-orbits of equations. . . *we just don't know which!*

The Commutative Algebra of Highly Symmetric Data

combinatorics

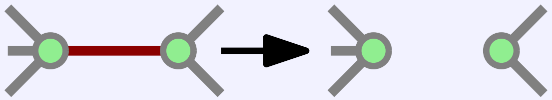
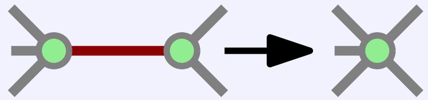
Conjecture

Over any field K, Sato's Grassmannian $\mathbf{Gr}_\infty(K)$ is Noetherian up to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N}) \subseteq \mathrm{GL}_\infty$.

The graph minor theorem

Graph minors

Any sequence of operations

 and 

takes a graph to a *minor*.

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Graph minors

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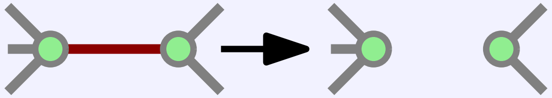
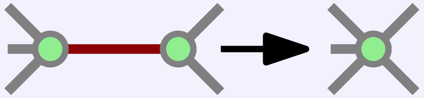
 and 

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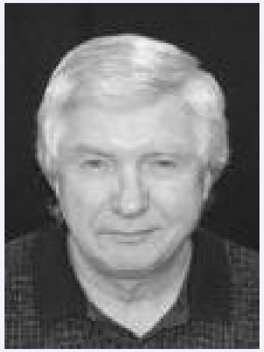
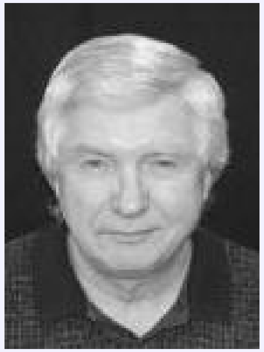


Robertson-Seymour [JCB 1983–2004, 669pp]

Any network property preserved under taking minors can be characterised by *finitely many forbidden minors*.

The graph minor theorem



Graph minors

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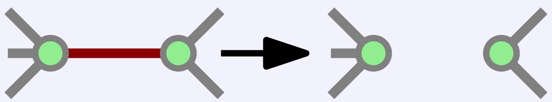
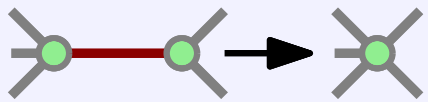
 and 

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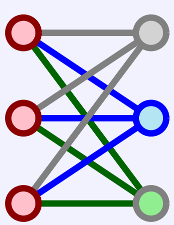
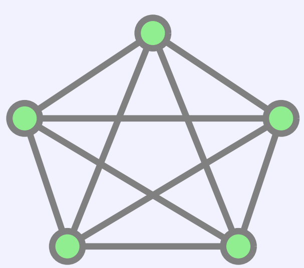
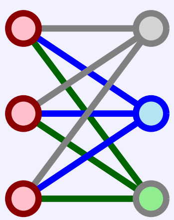
Wagner [Math Ann 1937]

For *planarity* these are

 and 

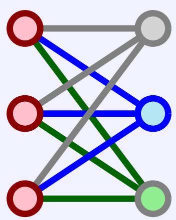
From graphs to matroids



1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
−1	−1	−1	0	0	0	0	0	0
0	0	0	−1	−1	−1	0	0	0
0	0	0	0	0	0	−1	−1	−1

column independence structure = matroid

From graphs to matroids



$$\begin{array}{ccccccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{array}$$

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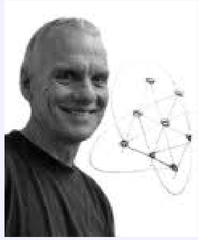

Matroid minor theorem (Geelen-Gerards-Whittle)

Any minor-preserved property of matroids over a fixed *finite field K* can be characterised by finitely many forbidden minors.

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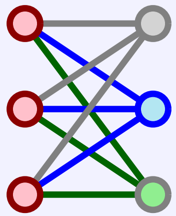

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

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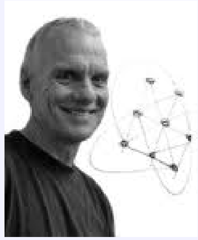

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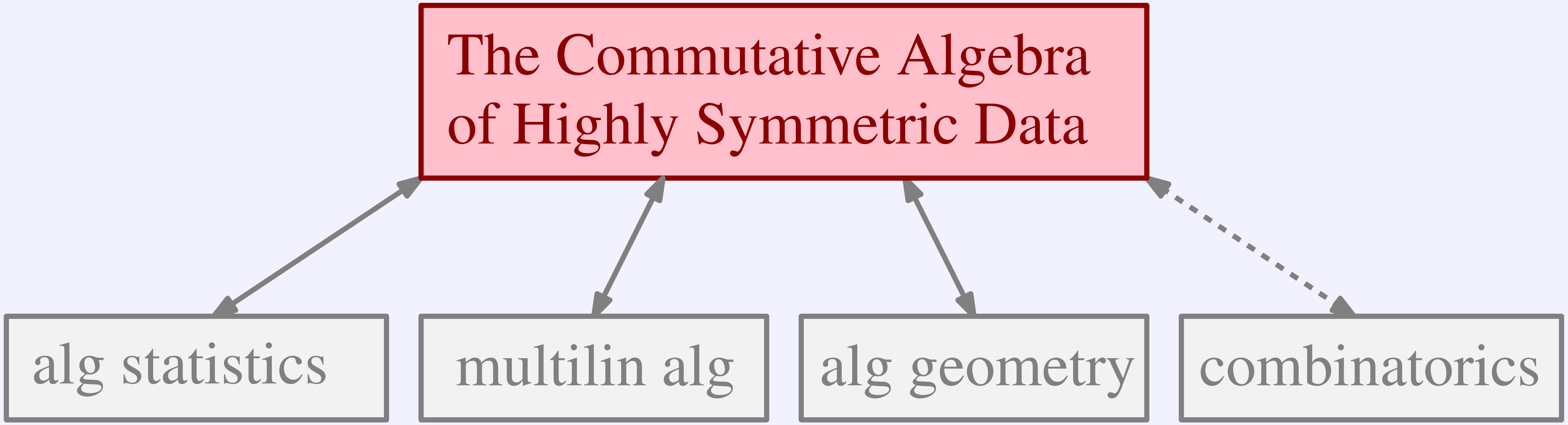
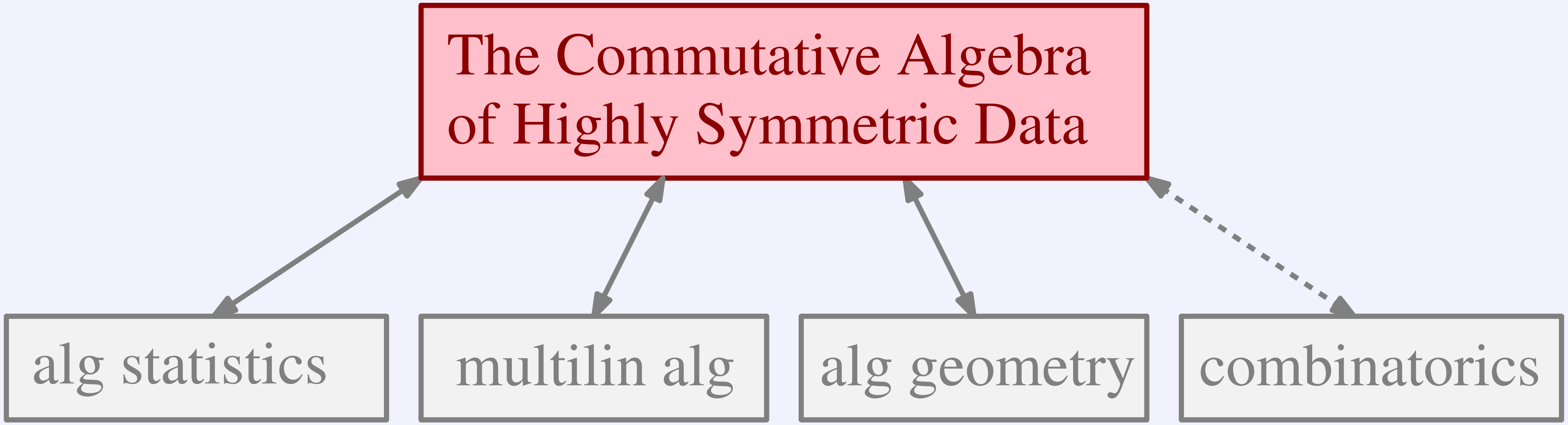
Surprising correspondence
Equivalant to $\mathrm{Sym}(-\mathbb{N} \cup +\mathbb{N})$-Noetherianity of $\mathbf{Gr}_\infty(K)$ (but Noetherianity may be true even for infinite K).



$\rightsquigarrow$ *theory and algorithms for highly symmetric, ∞-dim varieties*

$\rightsquigarrow$ *exciting interplay of algebra, combinatorics, statistics, and geometry*



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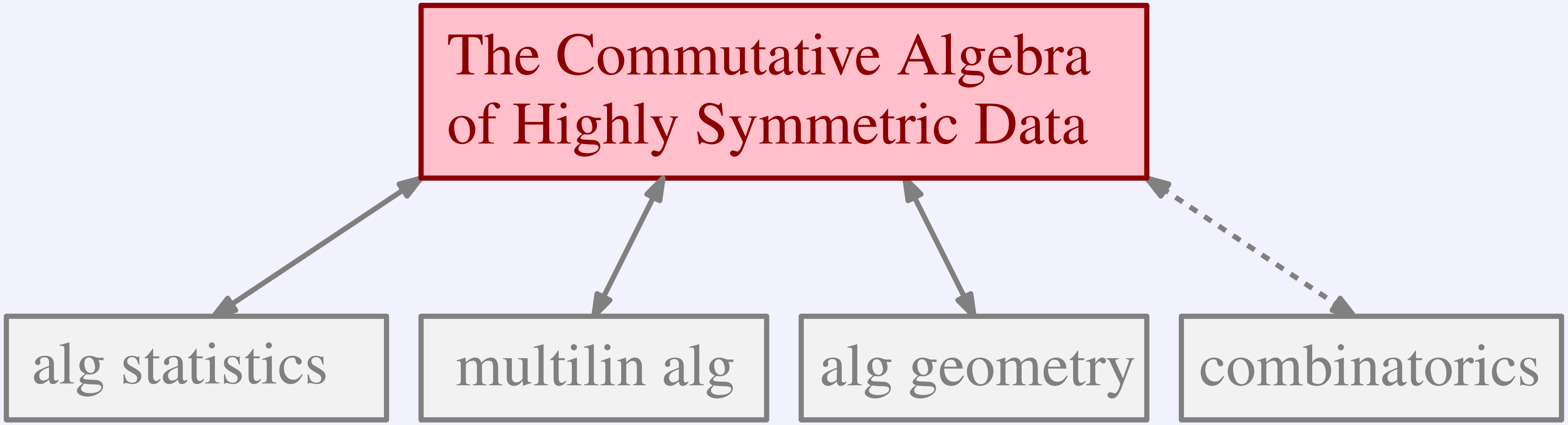

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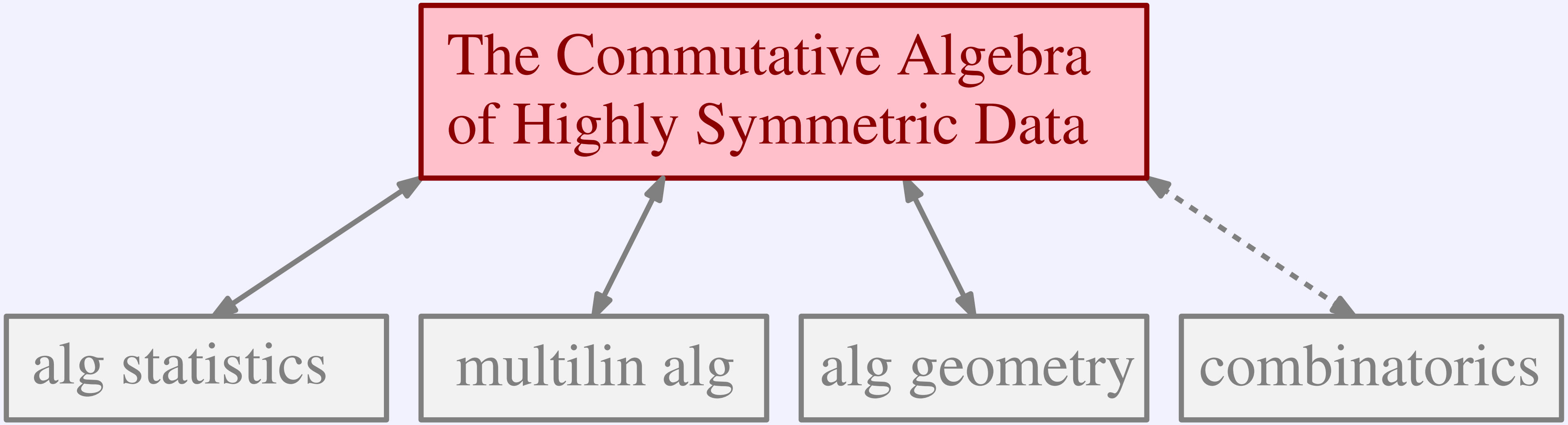
Paul Gordan

Ich habe mich davon überzeugt, daß
auch die Theologie ihre Vorzüge hat.



⇝ *theory and algorithms for highly symmetric, ∞-dim varieties*

⇝ *exciting interplay of algebra, combinatorics, statistics, and geometry*

Paul Gordan

Ich habe mich davon überzeugt, daß
auch die Theologie ihre Vorzüge hat.

Thank you!