# Tensor subrank: symmetric, border, and more

(Joint with Biaggi, Chang, Eggleston, Rupniewski, van Santen, Schnieper, Zuiddam, . . . )

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# Subrank



$V_1, \dots, V_d$ over $K$ of dims $n_1, \dots, n_d$, $T \in V_1 \otimes \cdots \otimes V_d$

**Definition** [Strassen]
The *subrank* $Q(T)$ of $T$ is the *maximal* $r$ s.t. $\exists$ linear $\varphi_i : V_i \to K^r$ with $\varphi_1 \otimes \cdots \otimes \varphi_d T = \sum_{i=1}^r e_i^{\otimes d} =: I_r$.

(For *rank* $R(T)$, the roles of $I_r$ and $T$ are swapped.)

**Remark**
The $\varphi_i$ are necessarily surjective, so $Q(T) \leq \min_i n_i$.

**Simple examples**
$Q(f) = 0 \Leftrightarrow f = 0$, $Q(I_r) = r$, $R(T) = 1 \Rightarrow Q(T) = 1$,
$Q(e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1) = 1$, and
$T \in (K^n)^{\otimes d}$ has $Q(T) = n \Leftrightarrow T \in \mathrm{GL}_n^d \cdot I_n$.

# Questions



- Generic subrank: $Q(T)$ for $T \in V_1 \otimes \cdots \otimes V_d$ sufficiently general? Almost equivalently: $\dim(\{T \mid Q(T) \geq r\}) =?$

**Definition**
The *border subrank* $\underline{Q}(T)$ of $T$ is the maximal $r$ such that $I_r \in \overline{\{\varphi_1 \otimes \cdots \otimes \varphi_d T \mid \varphi_i : V_i \to K^r\}}$. (Also $\leq \min_i n_i$.)

- Border version of the question above.

**Definition**
The *symmetric subrank* $Q_s(f)$ of $f \in S^d V$ is the maximal $r$ s.t. $\exists \varphi : V \to K^r$ with $(S^d \varphi) f = x_1^d + \cdots + x_r^d$.

- Generic symmetric (border) subrank?

- Behaviour of $Q$ under field extensions?

$L$ extension of $K \rightsquigarrow T_L \in (L \otimes_K V_1) \otimes_L \otimes \cdots \otimes_L (L \otimes_K V_n)$
$Q_L(T) := Q(T_L)$.

**Example**
$K = \mathbb{R}$ and $T =$
$e_1 \otimes e_1 \otimes e_1 - e_2 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2$
multiplication tensor of $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as an $\mathbb{R}$-algebra.

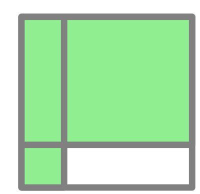
$\rightsquigarrow Q_{\mathbb{R}}(T) = 1 < 2 = Q_{\mathbb{C}}(T)$, because $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ as $\mathbb{C}$-algebras.

# 1. Generic subrank

**Theorem** [Derksen-Makam-Zuiddam]
For $K = \overline{K}$ and $V_1 = \cdots = V_d = K^n$ and $T$ sufficiently general, $Q(T) \approx C_d \cdot n^{1/(d-1)}$.

(Exact value was found by Pielasa-Šafránek-Shatsila!)

**Proof idea**
Write $K^n = K^r \oplus K^{n-r}$. $T$ with $Q(T) \geq r$ lies in the orbit of $I_r + T'$, where $T'$ lies in the space complementary to $(K^r)^{\otimes d}$. This space has stabiliser $P^d$, where $P =$



$r$ $n-r$

If such $T$ are dense in $(K^n) \otimes d$, then $r^d \leq dr(n-r)$. A construction shows that this also suffices ($\rightsquigarrow C_d = d^{1/(d-1)}$). $\square$

# 2. Generic border subrank



**Theorem** [Biaggi-Chang-D-Rupniewski]
For $K = \overline{K}$ and $V_1 = \cdots = V_d = K^n$ and $T$ sufficiently general, $\underline{Q}(T) \approx C_d \cdot n^{1/(d-1)}$.

But different $C_d$—indeed, $C_3 \geq 2 > 3^{1/2} \rightsquigarrow$ *general tensors have* $\underline{Q} > Q$.

I'll discuss instead:

**Theorem** [Chang-D-Schnieper-Zuiddam-…]
The locus $\{T \in (K^n)^{\otimes d} \mid \underline{Q}(T) = n\}$ is constructible of dimension $\approx \frac{d-1}{d} n^d$.

Note: $\underline{Q}(T) = n \Leftrightarrow I_n \in \overline{GT}$ where $G = \mathrm{GL}_n^d$.

# Generalised Hilbert-Mumford



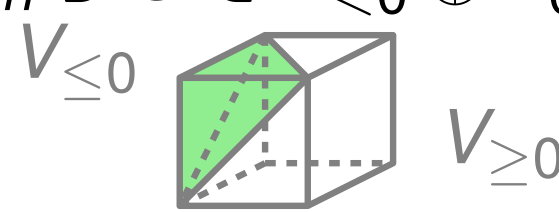
**Hilbert-Mumford criterion**
$G$ reductive acting on affine variety $X$; $p, q \in X$; $q \in \overline{Gp}$; *and* $Gq$ *closed* $\Rightarrow \exists \lambda : \mathbb{G}_{\mathbf{m}} \to G \; \exists q' \in Gq : \lim_{t \to 0} \lambda(t) p = q'$.

Would like to apply this with $X = (K^n)^{\otimes d} =: V$, $G = \mathrm{GL}_n^d$, $p = T$, $q = I_n$ ... *but* $G \cdot I_n$ *is not closed.*

**Generalised Hilbert-Mumford "criterion"** [BCDR]
$G$ reductive acting on affine variety $X$; $p, q \in X$; $q \in \overline{Gp}$; $\Rightarrow$ $\exists \lambda : \mathbb{G}_{\mathbf{m}} \to G \; \exists q' \in Gq : \lim_{t \to 0} \lambda(t) p = \lim_{t \to \infty} \lambda(t) q'$.

Applying this, decompose $V = V_{<0} \oplus V_0 \oplus V_{>0}$ using $\lambda$. Then $T \in V_0 \oplus V_{>0}$ and $G \cdot I_n \ni S \in V_{<0} \oplus V_0$ and $T_0 = S_0$.

Mental picture for $d = 3$:



# A combinatorial lemma

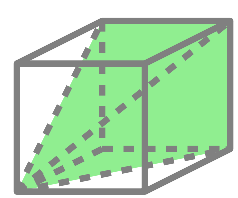


**Recall:** $G \cdot I_n \ni S \in V_{\leq 0}$. In particular, $S$ is $\mathrm{SL}_n^d$-*semistable*.
W.l.o.g. $\lambda_i(t) = \mathrm{diag}(t^{a_{i1}}, \dots, t^{a_{in}})$ for $i = 1, \dots, d$.
Set $D := \{(j_1, \dots, j_d) \mid a_{1,j_1} + \cdots + a_{d,j_d} \leq 0\} \subseteq [n]^d$
($\leftrightarrow$ a basis of $V_{\leq 0}$).

Semistability $\Rightarrow$ there is a probability distribution on $[n]^d$ supported in $D$ whose 1-dimensional margins are uniform.

**Lemma** [Claude]
$X_1, \dots, X_d$ rnd vars with $P(X_1 + \cdots + X_d \leq 0) = 1$, $Y_1, \dots, Y_d$ independent copies $\Rightarrow P(Y_1 + \cdots + Y_d \leq 0) \geq 1/d$.

$\rightsquigarrow \dim(V_{\leq 0}) \geq n^d/d$ and $\dim\{\underline{Q}(T) = n\} \leq n^d - n^d/d$.

Lower bound follows from:  $D$ $\square$

**Theorem** [Biaggi-D-de Nooij-van Santen]

For $K = \overline{K}$ and sufficiently general $f \in S^d K^n$, we have $Q_s(f) \approx (d! \cdot n)^{1/(d-1)}$.

**Proof sketch for $\geq$**

For $r \leq n$, consider sufficiently general $h_1, \ldots, h_{n-r} \in S^{d-1}K^r$. By [Hochser-Laksov, 1987], $\langle x_i \cdot h_j \mid i = 1, \ldots, r,\ j = 1, \ldots, n-r \rangle$ has dim $= \min\{r \cdot (n-r), \dim S^d K^r\}$.

$\rightsquigarrow$ If $r \cdot (n-r) \geq \dim S^d K^r$, then the tangent space to $\mathrm{GL}_n \cdot (x_1^d + \cdots + x_r^d + x_{r+1}h_1 + \cdots + x_n h_{n-r})$ projects surjectively to $S^d K^r$.

Left-hand side is essentially $\frac{n^d}{d!}$. $\square$

# 4. Comparing $Q_s$ and $\underline{Q}_s$.



**Theorem** [Biaggi-D-de Nooij-van Santen]

For $K = \overline{K}$ and sufficiently general $f \in S^d K^n$, we have $\underline{Q}_s(f) \approx C_d \cdot n^{1/(d-1)}$

We don't know whether $C_d > (d!)^{d-1}$; it's *harder* to degenerate to $x_1^d + \cdots + x_r^d$!

**Definition**

The Hilbert-Mumford subrank $Q_{\mathrm{HM}}(f)$ of $f \in S^d K^s$ is the maximal $r$ such that $\exists \lambda : \mathbb{G}_{\mathbf{m}} \to \mathrm{GL}_n$ with $\lim_{t \to 0} \lambda(t) f \in \mathrm{GL}_n \cdot (x_1^d + \cdots + x_r^d)$.

**Proposition** [BDdNvS]

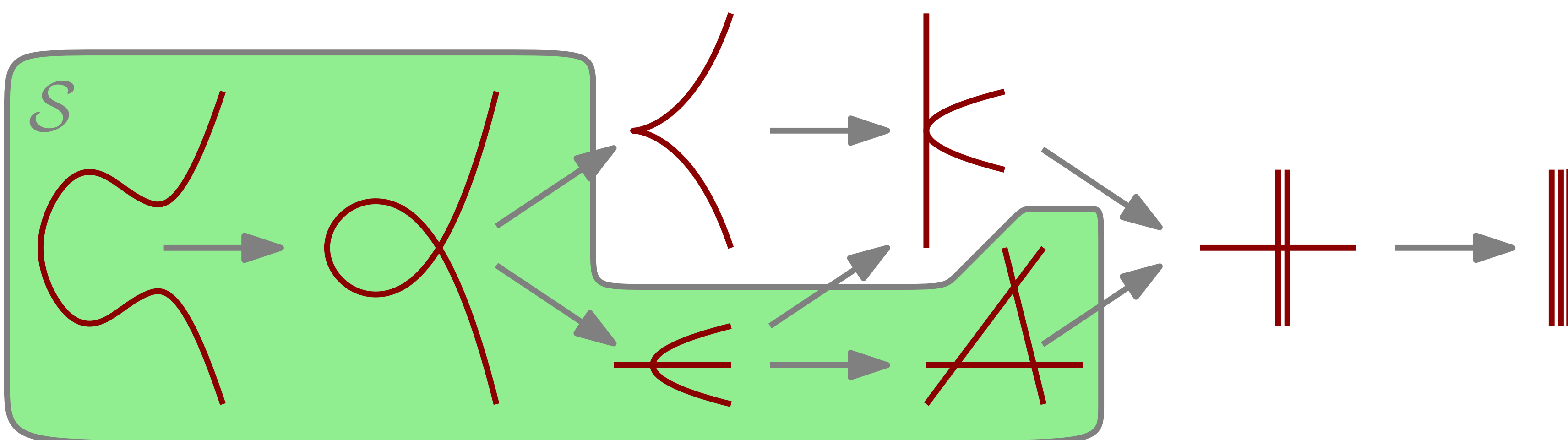
$Q_{\mathrm{HM}}(f) = Q_s(f)$. (Very different from ordinary tensors!)

# But can $Q_s$ be smaller than $\underline{Q}_s$?

**Theorem** ($K = \mathbb{C}$) [BDdNvS]
Assume $d = 3$ and $\underline{Q}_s(f) \leq 3$; then $Q_s(f) = \underline{Q}_s(f)$.

**Proof ingredients**

- Set $A := \mathbb{C}[x_1, x_2, x_3]_3$, so $\mathbb{P}A = \{\text{plane cubics}\}$.
- $\mathbb{C}[A]^{\mathrm{SL}_3}$ is generated by homogeneous polynomials $S$, $T$ of degrees 4 and 6, respectively.
- $\mathcal{S} = \{[h] \mid S(h) \neq 0 \text{ or } T(h) \neq 0\} = \{$cubics with at worst nodal singularities$\}$, the semistable locus in $\mathbb{P}A$.
- $\eta : \mathcal{S} = [S^3 : T^2] : \mathcal{S} \to \mathbb{P}^1$ quotient map.



# But can $Q_s$ be smaller than $\underline{Q}_s$?



Fix $f \in \mathbb{C}[x_0, x_1, \dots, x_3]_3$ defining a normal surface $X \subseteq \mathbb{P}^3$, set $\Pi := \{P \in \mathrm{Gr}(3,4) \mid P \cap X \in \mathcal{S}\}$, and $\pi : \Pi \to \mathbb{P}^1, P \mapsto \eta(P \cap X)$.

We show: either $f$ depends on $\leq 3$ variables, and $\mathrm{im}(\pi)$ is a point, or else $\pi$ is surjective, and in particular hits $\eta([x_1^3 + x_2^3 + x_3^3])$. $\square$

Similarly: $f$ of degree 4 and $\underline{Q}(f) \leq 2 \Rightarrow Q_s(f) = \underline{Q}(f)$.

**Example**

$f = u^2w^3 + v^5 + uvw^3 \in S^5\mathbb{C}^3$ has $Q_s(f) = 1 < 2 = \underline{Q}_s(f)$ (inspired by Shitov's example of $f$ with $Q_s(f) > Q(f)$).

# 5. Real versus complex subrank



Have seen $T$ with $Q_{\mathbb{R}}(T) = 1$, $Q_{\mathbb{C}}(T) = 2$, namely, scalar multiplication $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as real bilinear map.

Additivity of subrank holds for copies of $T$:

**Theorem** [Biaggi-D-Eggleston]
$\mathbb{R}$-rank of $T_n : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n, (a, b) \mapsto a * b = (a_i b_i)_i$ is $n$.

Note that $Q_{\mathbb{C}}(T_n) = 2n$.

**Theorem** [B-D-E]
For any real $T$ of order 3, we have $Q_{\mathbb{R}}(T) \geq \sqrt{Q_{\mathbb{C}}(T)}$.

**Example** [Zuiddam-ChatGPT Pro]
A $T \in \mathbb{R}^{2n} \otimes \mathbb{R}^{2n} \otimes \mathbb{R}^n$ with $Q_{\mathbb{R}}(T) \leq \sqrt{20 \cdot Q_{\mathbb{C}}(T)}$.

**Thank you!**

# References



• Benjamin Biaggi, Chia-Yu Chain, Jan Draisma, and Filip Rupniewski: *Border subrank via a generalised Hilbert-Mumford criterion*, Adv. Math. 461, Article ID 110077, 16 p. (2025).

• Benjamin Biaggi, Jan Draisma, and Sarah Eggleston: *Real subrank of order-three tensors*, arXiv:2503.17273

• Harm Derksen, Visu Makam, and Jeroen Zuiddam. *Subrank and optimal reduction of scalar multiplications to generic tensors.* J. Lond. Math. Soc., II. Ser., 110(2):26, 2024.