A short history of strength

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Matrix strength=rank

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or else $A = \sum_{i=1}^{k} u_i v_i^T, \quad k = \mathrm{rk}\,(A) < \min\{M, N\}.$

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$\rightsquigarrow$ either $\overline{\mathrm{GL}(U) \times \mathrm{GL}(V) A} = U \otimes V$

or else A is a polynomial in $2k$ *one*–tensors.

Matrix tuple strength=tuple rank

Definition. $A_1, \ldots, A_\ell \in \mathbb{C}^{N \times N} \rightsquigarrow \operatorname{rk}(A_1, \ldots, A_\ell) := \min\{\operatorname{rk}(c_1 A_1 + \cdots + c_\ell A_\ell) \mid (c_1 : \cdots : c_\ell) \in \mathbb{P}^{l-1}\}$

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Strength of cubics=q-rank

Definition (Derksen-Eggermont-Snowden)

$f = a_{111}x_1^3 + a_{112}x_1^2x_2 + \cdots + a_{ijk}x_ix_jx_k + \cdots$

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Theorem (Derksen, Eggermont, Snowden, 2017)

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Then f is a polynomial in $2k$ lower-degree forms.

Strength of higher-degree forms

Definition (Ananyan-Hochster)
strength($f = \sum_{i_1 \leq \cdots \leq i_d} c_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d}$) is
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Plan today: *generalise the dense/bounded dichotomy; applications; computational issues.*

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Examples of degree 3:
$PV = V \otimes V \otimes V, \quad PV = U \oplus V \oplus \bigwedge^2 V \oplus S^3 V.$

Polynomial transformations

$$\frac{\text{polynomial maps}}{\mathbb{A}^n} = \frac{?}{\text{polynomial functors}}$$

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Example (rank-one tensors). $PV = V \oplus V \oplus V$,
$QV = V^{\otimes 3}, \quad \alpha_V(v_1, v_2, v_3) = v_1 \otimes v_2 \otimes v_3.$

Polynomial transformations

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Classical
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Proposition (Bik-Eggermont-D-Snowden)
$X \subseteq P$ closed $\rightsquigarrow$ any polynomial transformation $X \to Q$ extends to P.

The dense/bounded dichotomy

Definition (lex order on polynomial functors)
P a poly functor with highest degree part $P_d \supseteq R \supsetneq 0$; $Q \prec P$ if Q a quotient of $V \mapsto P(U \oplus V)/R$.
Example: $S^2(U \oplus V)/S^2 V = S^2 U + U \otimes V$.

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Dichotomy (BDES) (*infinite version*)
Let $p \in \lim_{\leftarrow n} P(\mathbb{C}^n) =: P_\infty$. Then either $\mathrm{GL}_\infty p$ is dense in P_∞ or $\exists Q \prec P, \alpha : Q \to P, p \in \alpha(Q_\infty)$.

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Dichotomy (BDES) (*finite version*)
$X \subseteq P$ closed. Then either $X = P$ or $\exists Q_1, \ldots, Q_k \prec P$ and $\alpha_i : Q_i \to P$ such that $X \subseteq \bigcup \mathrm{im}\,(\alpha_i)$.

(*implies all earlier theorems*)

First applications

Theorem (D, 2017) (Noetherianity)
Any chain $P \supseteq X_1 \supseteq X_2 \supseteq \ldots$ (X_i closed) stabilises.

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Either all $X_m = P$ or some $X_{m_0} \subseteq \bigcup_{i=1}^{k} \operatorname{im} \alpha_i$ where $\alpha_i : Q_i \to P$ and $Q_i \prec P$. The pre-images $\alpha_i^{-1}(X_m), m \geq m_0$ stabilise by induction. $\square$

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Theorem (BDES) (orbit closures)
The map $\alpha \mapsto \overline{\operatorname{im} \alpha}$ is a surjection from $\{$polynomial transformations into $P\}$ to $\{$closures of $\mathrm{GL}_{\mathbb{N}}$-orbits in $P_\infty\}$.

The ultimate notion of strength?

Proposition/Definition (BDES)
For $p \in P_\infty$ there is a unique smallest Q such that $p \in \operatorname{im}\alpha$ for some $\alpha : Q \to P$. Call this Q the *strength* of p. (α *is unique mod* $\mathrm{Aut}_{\mathrm{poly}}(Q)$)

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Example: $P(V) = V \otimes V$.
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Dichotomy
$p \in P_\infty$ has a dense orbit or p has strength $\prec P$.

Stillman's conjecture

Theorem (Erman-Sam-Snowden 2018)
The ring R of bounded-degree series $\sum_{|\alpha|\leq d} c_\alpha x^\alpha$ in variables $x_1, x_2, \ldots$ *is a graded polynomial ring* in uncountably many variables of degrees $1, 2, 3, \ldots$

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Proof. Take $\{\ell_i\}_i, \{q_j\}_j, \{c_k\}_k, \ldots$ such that the q_j are a basis modulo the $\ell_i \ell_{i'}$, the c_k are a basis modulo the $\ell_i q_j$, etc. Any finite tuple has the expected strength $\rightsquigarrow$ are algebraically independent. $\square$

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Theorem (Ananyan-Hochster 2016, ESS2017)
There exists an upper bound $N(d_1, \ldots, d_k)$ on the projective dimension of $(f_1, \ldots, f_k)$ with $\deg(f_i) = d_i$.

A Gröbner proof of Stillman's conjecture

Theorem (D-Lasoń-Leykin, 2018)
Any finitely generated homogeneous ideal in R has a finite grevlex Gröbner basis w.r.t. $x_1 > x_2 > \dots$

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Theorem (D-Lasoń-Leykin, 2018)
There exists an algorithm that on input $d_1, \dots, d_k$ outputs all possible grevlex generic ideals of ideals $(f_1, \dots, f_k) \subseteq R$ where f_i homogeneous of degree d_i.

Final remarks

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