A tropical proof of the Brill-Noether theorem

Jan Draisma

Groningen, 19 March 2010

(j.w.w. Filip Cools, Elina Robeva, and Sam Payne)

The Brill-Noether theorem

X smooth projective curve of genus g
$D \in \operatorname{Div} X \rightsquigarrow |D| = \{E \geq 0 \mid D \sim E\}$; $\operatorname{rk}(D) := \dim |D|$

$d, r \in \mathbb{N}$
$W_d^r := \{[D] \in \operatorname{Pic}_d(X) \mid \operatorname{rk} D \geq r\}$
$\rho := g - (r+1)(g-d+r)$

Theorem
$\rho \geq 0 \Rightarrow W_d^r \neq \emptyset$ [Kempf 1971, Kleiman-Laksov 1972, Meis 1960, …]

$\rho < 0$ and X general $\Rightarrow W_d^r = \emptyset$
$\rho \geq 0$ and X general $\Rightarrow \dim W_d^r = \rho$ [Griffiths-Harris 1980]

$\rho = 0$ and X general $\Rightarrow |W_d^r| = \#$ standard tableaux of shape $(r+1) \times (g-d+r)$ with entries $1, 2, \ldots, g$

Why $\rho = g - (r+1)(g-d+r)$?

$$
\begin{array}{ccc}
TX^d & \xrightarrow{d\phi} & \phi^* T \operatorname{Pic}_d X \\
\downarrow & \swarrow & \\
X^d & \xrightarrow[\phi]{} & \operatorname{Pic}_d X \\
\bigcup & & \bigcup \\
W & \longrightarrow & W_d^r \\
|| & & \\
\{p \in X^d \mid \operatorname{rk} d_p\phi \leq d - r\} & &
\end{array}
$$

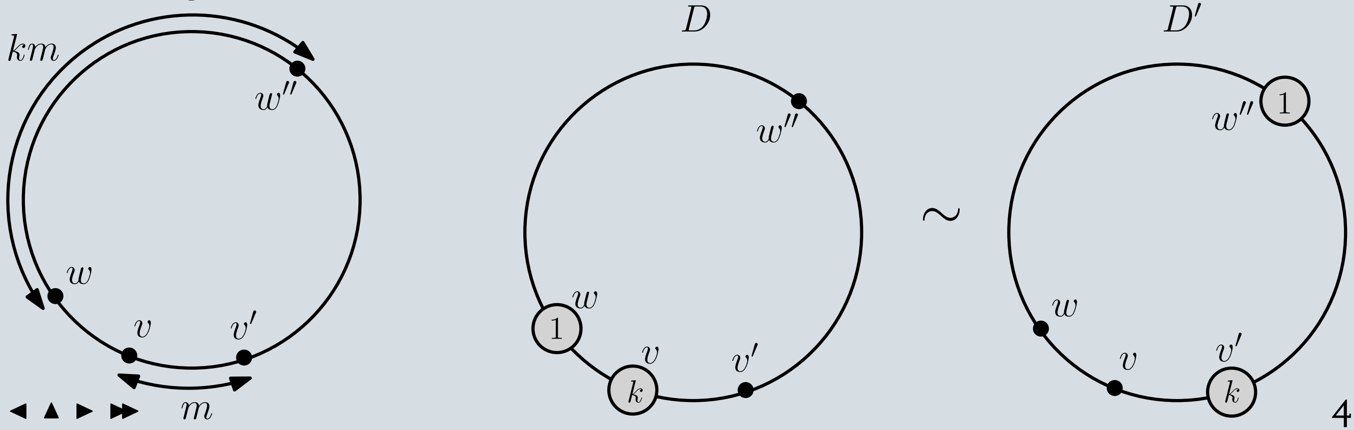
$\rightsquigarrow$ expected $\dim W = d - r(g - d + r)$

$\rightsquigarrow$ expected $\dim W_d^r = d - r(g - d + r) - r = \rho$

Linear systems on metric graphs

[Baker-Norine, Gathmann-Kerber, Mikhalkin-Zharkov]
Γ metric graph
$\operatorname{Div}\Gamma := \mathbb{Z}\Gamma$
$M(\Gamma) :=$ {piecewise linear functions $\Gamma \to \mathbb{R}$ with $\mathbb{Z}$-slopes}
$\operatorname{div} f := \sum_{v\in\Gamma} \operatorname{ord}_v(f)v \sim 0$ principal divisors
$|D| := \{E \geq 0 \mid E \sim D\}$
$\operatorname{rk} D := \max\{r \mid |D - v_1 - \ldots - v_r| \neq \emptyset$ for all $v_i \in \Gamma\}$

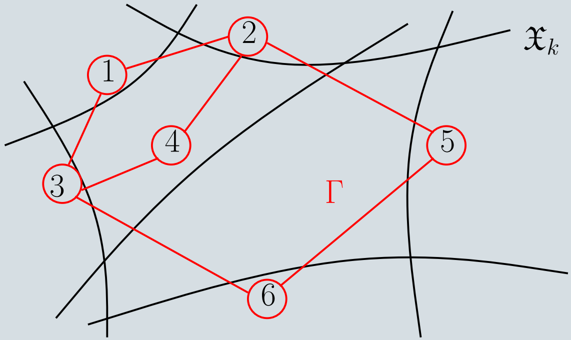
Chip-firing interpretation



Specialization Lemma (Baker, 2007)

K discretely valued field, $R \subseteq K$ valuation ring, k residue field
X smooth curve over K

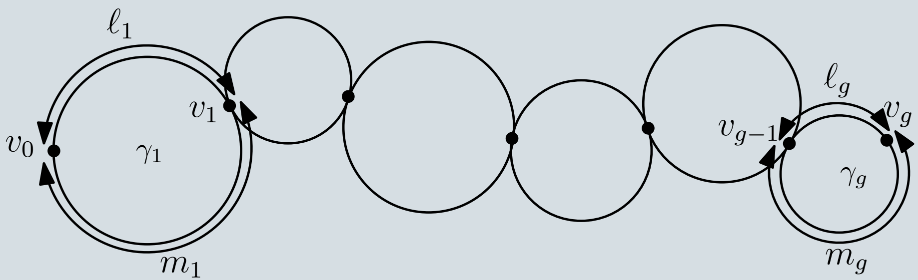
Strongly semistable model of X
$\mathfrak{X}$ proper, regular, flat scheme over $\operatorname{Spec} R$
general fibre $\mathfrak{X}_K$ isomorphic to X
special fibre $\mathfrak{X}_k = X_1 \cup \ldots \cup X_s$
intersections simple nodes $/k$
$\rightsquigarrow$ dual graph Γ on $\{u_1, \ldots, u_s\}$
(metric with edge lengths 1)
$\rightsquigarrow$ map $X(K) = \mathfrak{X}(R) \to \{u_1, \ldots, u_s\}$



well-behaved with respect to finite extensions K'/K
$\rightsquigarrow$ specialisation map $\tau : X(\overline{K}) = \mathfrak{X}(R) \to \Gamma$

$$D \in \operatorname{Div}_d(X_{\overline{K}}) \Rightarrow \operatorname{rk}(\tau_* D) \geq \operatorname{rk}(D)$$

A Brill-Noether general Γ_g



$d, r \in \mathbb{N}$, $\rho := g - (r+1)(g-d+r)$
$W_d^r := \{[D] \in \mathrm{Pic}_d(\Gamma_g) \mid \mathrm{rk}(D) \geq r\}$

Main Theorem
$\rho < 0 \Rightarrow W_d^r = \emptyset$
$\rho \geq 0 \Rightarrow \dim W_d^r = \rho$
$\rho = 0 \Rightarrow \#W_d^r = \#$ standard tableaux of shape $(r+1) \times (g-d+r)$ with entries $1, 2, \ldots, g$

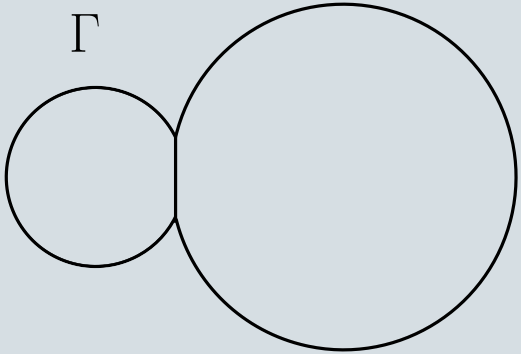
$\Rightarrow$ G-H 1980! [Specialization and Conrad's appendix to Baker 2007]

The BN Game

  or  

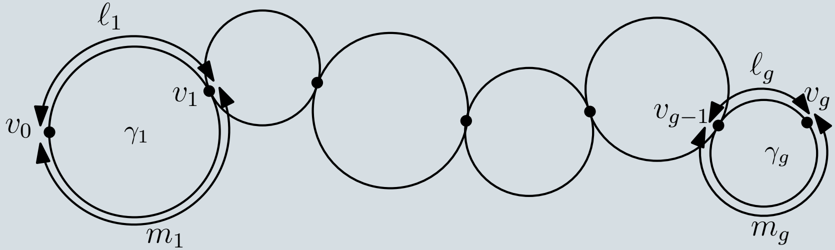
Rules

B puts d chips on a metric graph
(i.e., chooses $D \geq 0, \ \mathrm{rk}\, D = d$)
N challenges by specifying r positions
B wins if he can fire to cover those
N wins otherwise



$d = 2, r = 1$—who wins?

Analysing the BN game on Γ_g



may assume that N's positions are in $\{v_0, \ldots, v_g\}$ [Luo]

may assume that D has d_0 chips at v_0 and ≤ 1 chip on each γ_i

$D \rightsquigarrow$ **lingering lattice path** $P : p_0, p_1, \ldots, p_g \in \mathbb{Z}^r$

$p_0 := (d_0, d_0 - 1, \ldots, d_0 - r + 1)$

$$p_i - p_{i-1} = \begin{cases} (-1, \ldots, -1) & \text{if B has no chip on } \gamma_i \\ e_j & \text{if firing } p_{i-1}(j) \text{ chips from } v_{i-1} \text{ to } v_i \\ & \text{leads the chip on } \gamma_i \text{ to } v_i \text{ as well} \\ & \textbf{and } p_{j-1}, p_{j-1} + e_j \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{C} := \{(y(0), \ldots, y(r-1)) \in \mathbb{Z}^r \mid y(0) > y(1) > \ldots > y(r-1) > 0\}$

Analysis, continued

Proposition
B wins with starting position D iff P stays entirely in $\mathcal{C}$.

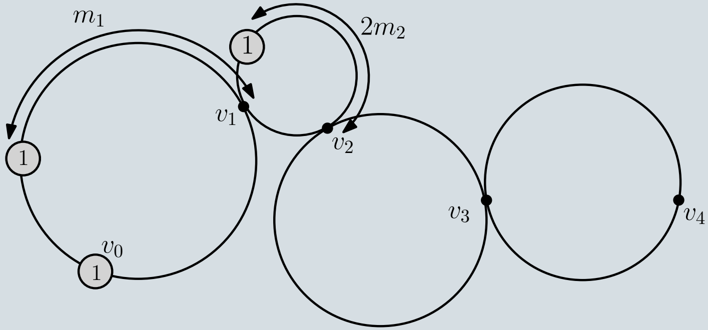
Proposition $\Rightarrow$ Main Theorem
assume $W_d^r \neq \emptyset$
P lingering lattice path with a winning D
$d_0 \geq r$
$\#$ steps in direction $(-1, \ldots, -1) = g - d + d_0$
$0 < p_g(r-1) = (d_0 - r + 1) - (g - d + d_0) + \#$ steps in direction e_{r-1}
$\rightsquigarrow \#$ steps in direction $e_{r-1} \geq g - d + r$
$\rightsquigarrow \#$ steps in direction $e_i \geq g - d + r$, all i
$\rightsquigarrow d - d_0 \geq r(g - d + r)$
$\rightsquigarrow d - r \geq r(g - d + r)$
$\rightsquigarrow g \geq (r+1)(g - d + r)$ $\square$

Example

$g = 4, d = 3, r = 1$
$\rightsquigarrow \rho = 0$

1	3
2	4

$\rightsquigarrow 1, 2, 3, 2, 1$



1	2
3	4

$\rightsquigarrow 1, 2, 1, 2, 1$