(Uniform) determinantal representations

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October 2016, Kolloquium Bern

Determinantal representations

$R := \mathbb{C}[x_1, \ldots, x_n]$ and $R_{\leq d} := \{p \in R \mid \deg p \leq d\}$

Definition

A *determinantal representation* of $p \in R$ of *size* N is a matrix $M \in R_{\leq 1}^{N \times N}$ with $\det(M) = p$.

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$n = 1$**: companion matrices**

$$\det \begin{bmatrix} x & -1 & & & \\ & x & -1 & & \\ & & \ddots & \ddots & \\ & & & x & -1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} + a_n x \end{bmatrix} = a_0 + a_1 x + \ldots + a_n x^n$$

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A bivariate example

$$\det \begin{bmatrix} x & -1 & \\ y & & -1 \\ a + bx + cy & dx + ey & fy \end{bmatrix} = a + bx + cy + dx^2 + exy + fy^2$$

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Determinantal representations always exist, but how small?
⇝ the *determinantal complexity* $\mathrm{dc}(p)$ is the smallest N.

Why?

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"If p has a determinantal representation M of small size N, then p can be evaluated efficiently using Gaussian elimination."

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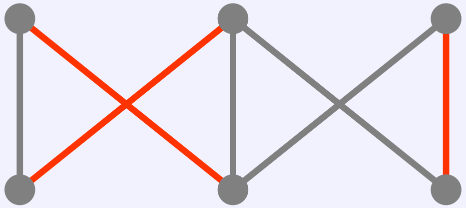
Definition

$\mathrm{perm}_m := \sum_{\pi \in S_m} x_{1\pi(1)} \cdots x_{m\pi(m)}$ is the $m \times m$ *permanent*.

Example

$$\mathrm{perm}_3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3$$ counts *perfect matchings*:



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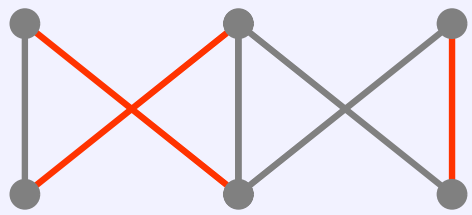

Counting matchings in bipartite graphs is believed hard, so $\mathrm{dc}(\mathrm{perm}_m)$ should be large!

Geometric complexity theory

Conjecture [Valiant, 70s]

$\mathrm{dc}(\mathrm{perm}_m)$ grows faster with m than any polynomial.

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Best known bounds [Mignon-Ressayre 04, Grenet 12]

$\frac{m^2}{2} \leq \mathrm{dc}(\mathrm{perm}_m) \leq 2^m - 1$ [Alper-Bogart-Velasco 15: $= 7$ for $m = 3$]

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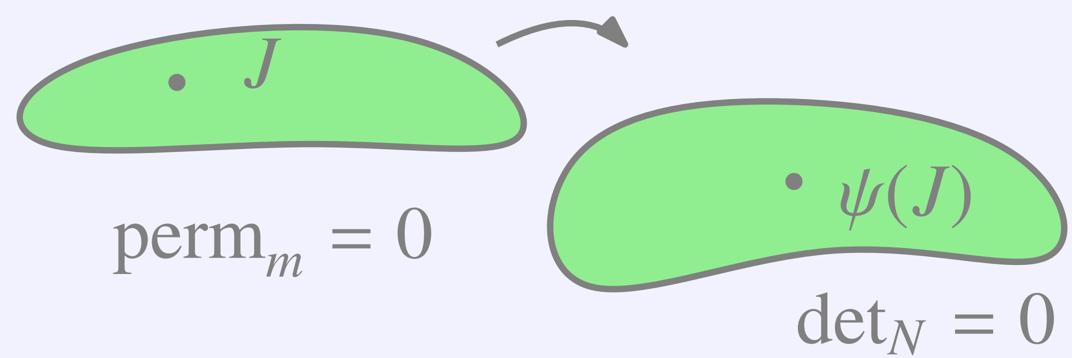
Proof sketch of lower bound

If $\psi : \mathbb{C}^{m\times m} \to \mathbb{C}^{N\times N}$ affine-linear with $\det_N(\psi(A)) = \mathrm{perm}_m(A)$,

$$J := \begin{bmatrix} -m+1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$



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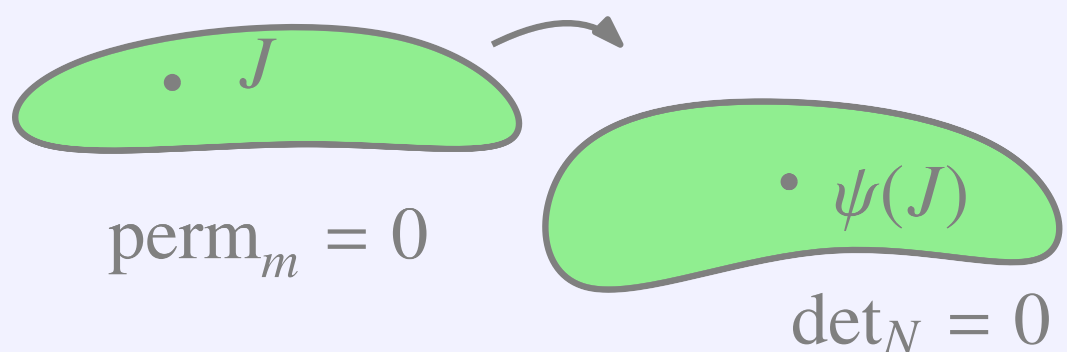

$q_1(X) :=$ quadratic part of $\mathrm{perm}_m(J + X)$, form of rank m^2

$q_2(Y) :=$ quadratic part of $\det_N(\psi(J) + Y)$, form of rank $\leq 2N$

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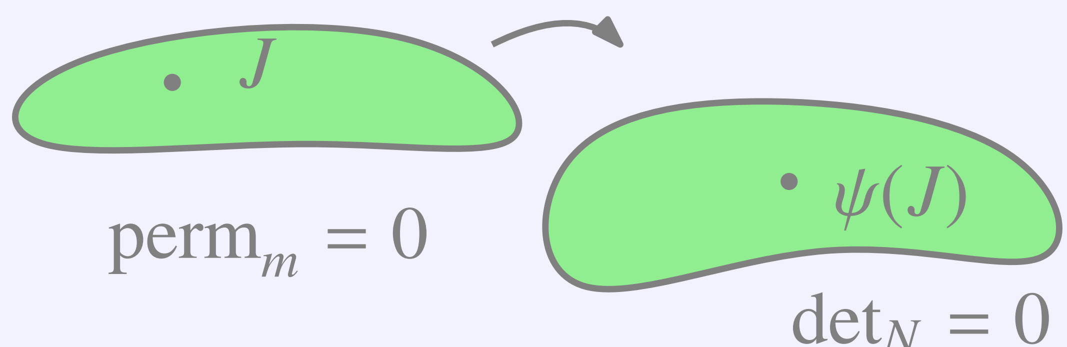

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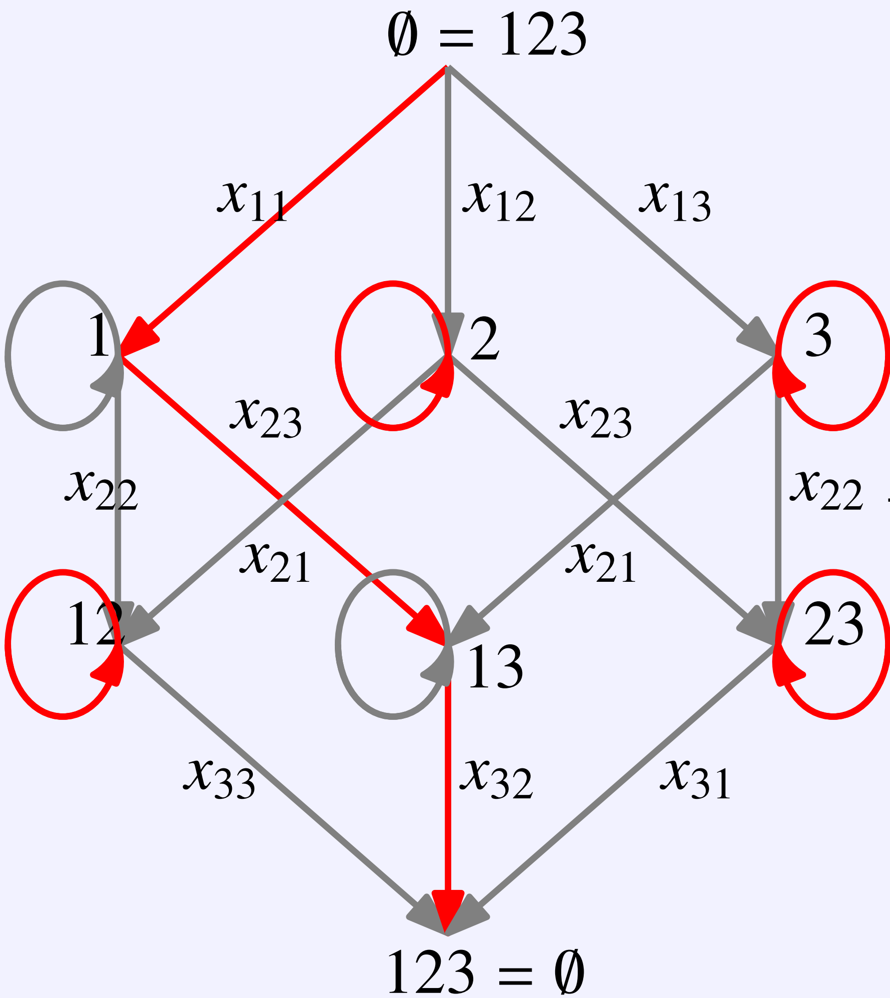
Now $q_1(X) = q_2(L(X))$ where L linear part of ψ, so $m^2 \leq 2N$. $\square$

Grenet's $2^m - 1$ construction



	∅ = 123	1	2	3	12	13	23
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1	x_{11}	1	0	0	0	0	0
2	x_{12}	0	1	0	0	0	0
3	x_{13}	0	0	1	0	0	0
12	0	x_{22}	x_{21}	0	1	0	0
13	0	x_{23}	0	x_{21}	0	1	0
23	0	0	x_{23}	x_{22}	0	0	1

x_{ij} labels an arrow from an $(i-1)$-set to an i-set by adding j.

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Compare orbit closures X_1, X_2 of $\ell^{N-m}\mathrm{perm}_m$ and $\det_N$ inside the space of degree-N polynomials in N^2 variables under $G = \mathrm{GL}_{N^2}$; try to show that $X_1 \not\subseteq X_2$ by showing that multiplicities of certain G-representations are higher in $\mathbb{C}[X_1]$ than in $\mathbb{C}[X_2]$ unless N is super-polynomial in m.

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Theorem [Bürgisser-Ikenmeyer-Panova, 16]
This approach does not work if *higher than* is restricted to $1 > 0$ (so-called *occurrence obstructions*).

Motivation II: Solving systems of equations

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Proposal [Plestenjak-Hochstenbach, 16]

To solve $p(x, y) = q(x, y) = 0$ write $p = \det(A_0 + xA_1 + yA_2)$ and $q = \det(B_0 + xB_1 x + yB_2)$ and solve the *two-parameter eigenvalue problem* $(A_0 + xA_1 + yA_2)u = 0$ and $(B_0 + xB_1 + yB_2)v = 0$.

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$\rightsquigarrow$ translates to a joint pair of *generalised eigenvalue problems:* $(\Delta_1 - x\Delta_0)w = 0$ and $(\Delta_2 - y\Delta_0)w = 0$ where $w = u \otimes v$ and $\Delta_0 = A_1 \otimes B_2 - A_2 \otimes B_1$, $\Delta_1 = A_2 \otimes B_0 - A_0 \otimes B_2$, $\Delta_2 = A_0 \otimes B_1 - A_1 \otimes B_0$

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If the sizes are N, then Δ_i have size N^2, and solving takes $(N^2)^3 \ldots$ (plane curves have det rep of size = deg, but harder to compute).

Determinantal complexity of general polynomials

Theorem [Boralevi-v Doornmalen-D-Hochstenbach-Plestenjak, 16]
For n fixed, there exist C_1, C_2 such that a *sufficiently general* $p \in R_{\leq d}$ has $\mathrm{dc}(p) \geq C_1 d^{n/2}$ and *any* $p \in R_{\leq d}$ has $\mathrm{dc}(p) \leq C_2 d^{n/2}$.

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Proof of lower bound
If sufficiently general $p \in R_{\leq d}$ have $\mathrm{dc}(p) \leq N$, then the map $\det : R_{\leq 1}^{N \times N} \to R_{\leq N}$ contains $R_{\leq d}$ in the closure of its image. Comparing dimensions, find $N^2 \cdot (n+1) \geq \dim_{\mathbb{C}} R_{\leq d} = \binom{n+d}{n}$. □

Spaces connected to 1

Definition

Given a nonzero subspace $V \subseteq R$ write $V_{\leq d} := V \cap R_{\leq d}$. V is *connected to* 1 if $V_{\leq d+1} \subseteq R_{\leq 1} \cdot V_{\leq d}$ for all $d \geq 0$.

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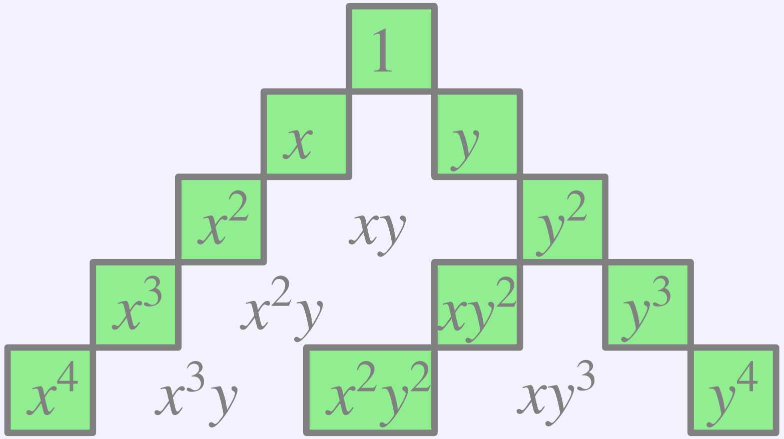
Example
For $n = 2$, V spanned by these monomials is connected to 1:



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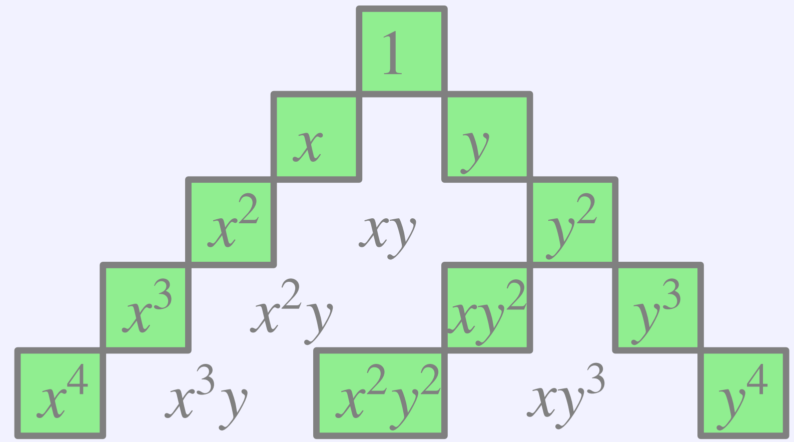

Lemma

V connected to 1, with basis $1 = f_1, f_2, \ldots, f_m$ of ascending degrees, write $f_i = \sum_{j=1}^{i-1} \ell_{ij} f_j$ with $\ell_{ij} \in R_{\leq 1}$. Then $V =$ the span of the $(m-1)\times(m-1)$-subdeterminants of

$$M(V) := \begin{bmatrix} \ell_{21} & -1 & & & \\ \ell_{31} & \ell_{32} & -1 & & \\ \vdots & & \ddots & \ddots & \\ \ell_{m1} & \ell_{m2} & \cdots & \ell_{m,m-1} & -1 \end{bmatrix}$$

Proposition

Let $V \subseteq R$ be connected to 1, of dimension m, and such that $R_{\leq 1} \cdot V \supseteq R_{\leq d}$. Then there is a uniform determinantal representation of size m for the polynomials in $R_{\leq d}$.

First construction

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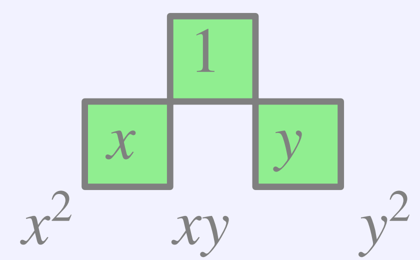

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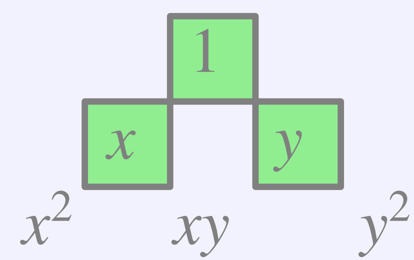

Theorem [Hochstenbach-Plestenjak 16]
For $n = 2$ there exist uniform det representations of size $\sim \frac{d^2}{4}$.

Analysis of first construction

V connected to 1 and $R_{\leq 1} \cdot V \supseteq R_{\leq d}$ imply $\dim V \geq \frac{1}{n}\binom{n+d}{n}$

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Construction uses the lattice of type A_{n-1} with generating matrix

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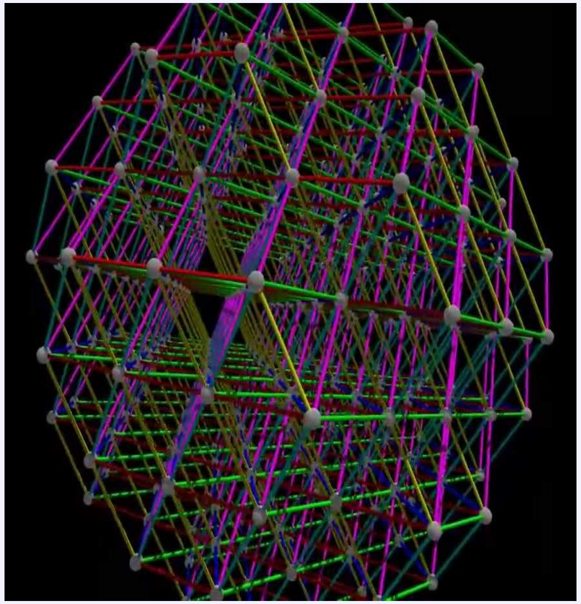

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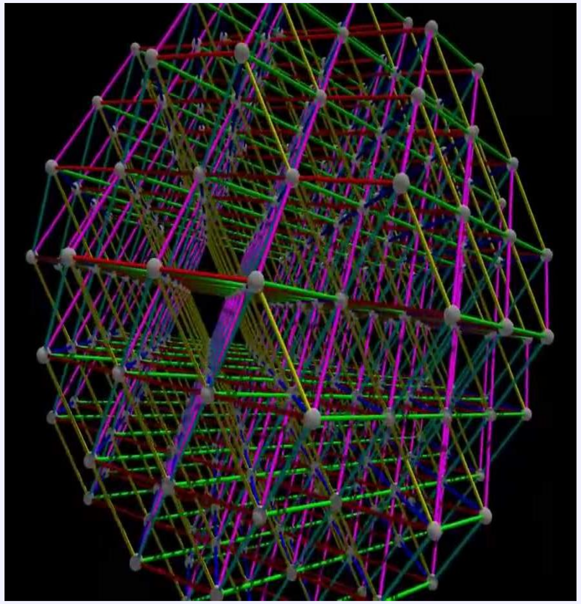

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But the exponent of d is n rather than $n/2$.

Second construction: divide and conquer!

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$M(V_1)$ (upper-left block), $M(V_2)^T$ (lower-right block)

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Can we find V_1, V_2, connected to 1, of dim $\sim \sqrt{\dim R_{\leq d}}$ such that $(R_1 \cdot) V_1 \cdot V_2 \supseteq R_{\leq d}$?

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- start with $B_0 := \sum'^{\infty}_{j=0}\{0, 1\} \cdot 2^{2j}$;
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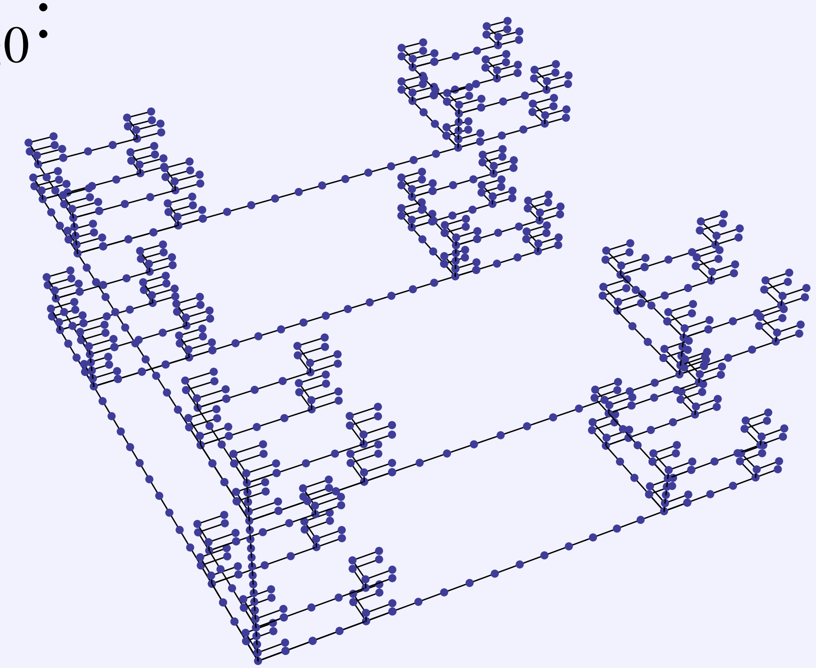

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- start with $B_0 := \sum'^{\infty}_{j=0}\{0,1\} \cdot 2^{2j}$;
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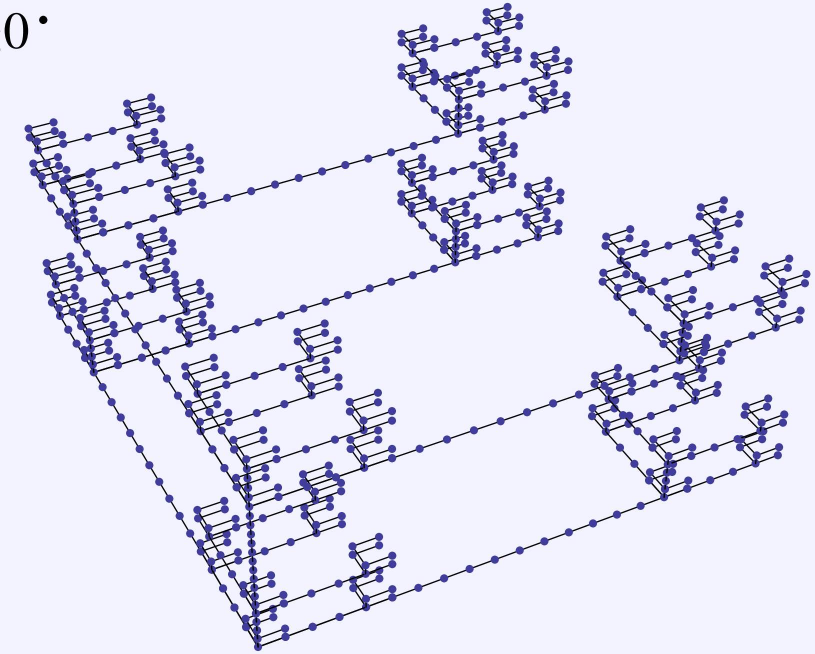
- $A_i := B_i^n$;
- connect to 0.



Take V_i spanned by the monomials with exponent vectors in A_i. $\square$

Outlook

Theorem [Boralevi-v Doornmalen-D-Hochstenbach-Plestenjak, 16]
For n fixed, there exist C_1, C_2 such that a *sufficiently general* $p \in R_{\leq d}$ has $\mathrm{dc}(p) \geq C_1 d^{n/2}$ and *any* $p \in R_{\leq d}$ has $\mathrm{dc}(p) \leq C_2 d^{n/2}$.

Many questions remain:

- what are the best constants C_1, C_2?
- what about the regime where d is fixed and n runs?
- *symmetric* determinantal representations?

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Motivation III: hyperbolic polynomials
If $p = \det(A_0 + \sum_i x_i A_i)$ with $A_i \in \mathbb{R}^{N \times N}$ symmetric and A_0 positive definite, then the restriction of p to any line through 0 has only real roots. For $n = 2$ the converse also holds (Helton-Vinnikov).